Question 1. **1 mark for writing in full sentences.**
1 mark for at least two differences.
1 mark for at least two similarities.
1 mark for a first-order Laplace example. (Question 6 is acceptable.)
1 mark for a first-order non-Laplace example. (Questions 4 or 5 are acceptable.)

Question 2. $y'' + \omega^2 y = \delta\left(t - \frac{\pi}{\omega}\right)$, $y(0) = 1$, $y'(0) = 0$.

Taking the Laplace transform, we have

$$\begin{aligned}
s^2\mathcal{L}[y] - sy(0) - y'(0) + \omega^2\mathcal{L}[y] &= e^{-\pi s/\omega} \\
(s^2 + \omega^2)\mathcal{L}[y] &= e^{-\pi s/\omega} + s \\
\mathcal{L}[y] &= \frac{e^{-\pi s/\omega}}{s^2 + \omega^2} + \frac{s}{s^2 + \omega^2} \\
y &= \frac{1}{\omega} u_{\pi/\omega}(t) \sin\left(\omega\left(t - \frac{\pi}{\omega}\right)\right) + \cos\omega t \\
&= \frac{1}{\omega} u_{\pi/\omega}(t) \sin(\omega t - \pi) + \cos\omega t
\end{aligned}$$

Version A: $\omega = 3$, so $y = \frac{1}{3}u_{\pi/3}(t)\sin(3t - \pi) + \cos 3t$.
Version B: $\omega = 4$, so $y = \frac{1}{4}u_{\pi/4}(t)\sin(4t - \pi) + \cos 4t$.
Version C: $\omega = 5$, so $y = \frac{1}{5}u_{\pi/5}(t)\sin(5t - \pi) + \cos 5t$.

[1 mark for taking the Laplace transform, 1 mark for finding $\mathcal{L}[y]$, 2 marks for the first inverse transform, 1 mark for the second inverse transform]

Question 3. Version A. $y'' - y' - 6y = 0$, $y(0) = 2$, $y'(0) = 3$

Taking the Laplace transform, we have

$$\begin{aligned}
s^2\mathcal{L}[y] - sy(0) - y'(0) - s\mathcal{L}[y] + y(0) - 6\mathcal{L}[y] &= 0\\
(s^2 - s - 6)\mathcal{L}[y] &= 2s + 1\\
\mathcal{L}[y] &= \frac{2s+1}{(s+2)(s-3)}
\end{aligned}$$

Using partial fractions, we have

$$\begin{aligned}
\frac{2s+1}{(s+2)(s-3)} &= \frac{A}{s+2} + \frac{B}{s-3}\\
2s + 1 &= A(s-3) + B(s+2)
\end{aligned}$$

$$\begin{aligned}
&s = 3 & 7 &= 5B & B &= \frac{7}{5}\\
&s = -2 & -3 &= -5A & A &= \frac{3}{5}
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{L}[y] &= \frac{1}{5}\left(\frac{3}{s+2} + \frac{7}{s-3}\right)\\
y &= \frac{3}{5}e^{-2t} + \frac{7}{5}e^{3t}
\end{aligned}$$

[1 mark for taking the Laplace transform, 1 mark for finding $\mathcal{L}[y]$, 2 marks for the partial fractions, 1 mark for the inverse transform]

Question 3. Version B. $y'' - y' - 6y = 0$, $y(0) = 4$, $y'(0) = 5$

Taking the Laplace transform, we have

$$\begin{aligned} s^2\mathcal{L}[y] - sy(0) - y'(0) - s\mathcal{L}[y] + y(0) - 6\mathcal{L}[y] &= 0 \\ (s^2 - s - 6)\mathcal{L}[y] &= 4s + 1 \\ \mathcal{L}[y] &= \frac{4s+1}{(s+2)(s-3)} \end{aligned}$$

Using partial fractions, we have

$$\begin{aligned} \frac{4s+1}{(s+2)(s-3)} &= \frac{A}{s+2} + \frac{B}{s-3} \\ 4s + 1 &= A(s-3) + B(s+2) \end{aligned}$$

$$\begin{array}{lcr} s = 3 & 13 = 5B & B = \dfrac{13}{5} \\ s = -2 & -7 = -5A & A = \dfrac{7}{5} \end{array}$$

Hence we have

$$\begin{aligned} \mathcal{L}[y] &= \frac{1}{5}\left(\frac{7}{s+2} + \frac{13}{s-3}\right) \\ y &= \frac{7}{5}e^{-2t} + \frac{13}{5}e^{3t} \end{aligned}$$

[1 mark for taking the Laplace transform, 1 mark for finding $\mathcal{L}[y]$, 2 marks for the partial fractions, 1 mark for the inverse transform]

Question 3. Version C. $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = -1$

Taking the Laplace transform, we have

$$\begin{aligned} s^2\mathcal{L}[y] - sy(0) - y'(0) - s\mathcal{L}[y] + y(0) - 6\mathcal{L}[y] &= 0 \\ (s^2 - s - 6)\mathcal{L}[y] &= s - 2 \\ \mathcal{L}[y] &= \frac{s-2}{(s+2)(s-3)} \end{aligned}$$

Using partial fractions, we have

$$\begin{aligned} \frac{s-2}{(s+2)(s-3)} &= \frac{A}{s+2} + \frac{B}{s-3} \\ s - 2 &= A(s-3) + B(s+2) \end{aligned}$$

$$\begin{aligned} s &= 3 & 1 &= 5B & B &= \frac{1}{5} \\ s &= -2 & -4 &= -5A & A &= \frac{4}{5} \end{aligned}$$

Hence we have

$$\begin{aligned} \mathcal{L}[y] &= \frac{1}{5}\left(\frac{4}{s+2} + \frac{1}{s-3}\right) \\ y &= \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t} \end{aligned}$$

[1 mark for taking the Laplace transform, 1 mark for finding $\mathcal{L}[y]$, 2 marks for the partial fractions, 1 mark for the inverse transform]

Question 4. Version A. $3xy + y^2 + (x^2 + xy)y' = 0$, $y(1) = -6$

If $\tilde{M} = 3xy + y^2$ and $\tilde{N} = x^2 + xy$, then

$$\frac{\partial \tilde{M}}{\partial y} = 3x + 2y \qquad \frac{\partial \tilde{N}}{\partial x} = 2x + y$$

This equation is not exact. However, we can use an integrating factor:

$$\begin{aligned} \frac{\frac{\partial \tilde{M}}{\partial y} - \frac{\partial \tilde{N}}{\partial x}}{\tilde{N}} &= \frac{x+y}{x^2+xy} = \frac{1}{x} \\ \mu &= e^{\int x^{-1}dx} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

We can thus rewrite the equation as $(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$.

If $M = 3x^2y + xy^2$ and $N = x^3 + x^2y$, then

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy \qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

Hence the equation is exact. Let

$$\begin{aligned} F(x,y) &= \int M(x,y)dx + g(y) \\ &= \int (3x^2y + xy^2)dx + g(y) \\ &= x^3y + \frac{1}{2}x^2y^2 + g(y) \\ \frac{\partial F}{\partial y} &= x^3 + x^2y + g'(y) = N = x^3 + x^2y \\ \therefore\ g'(y) &= 0 \\ g(y) &= k \\ F(x,y) &= x^3y + \frac{1}{2}x^2y^2 = C \text{ (where the constant } k \text{ has been absorbed)} \end{aligned}$$

We can find the solution since we have a quadratic in y:

$$\begin{aligned} \frac{1}{2}x^2y^2 + x^3y - C &= 0 \\ y &= \frac{-x^3 \pm \sqrt{x^6 + 2x^2C}}{x^2} \end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned} 18 - 6 - C &= 0 \\ C &= 12 \end{aligned}$$

Only the negative root corresponds to this initial condition. Hence the solution is

$$y = \frac{-x^3 - \sqrt{x^6 + 24x^2}}{x^2}$$

[1 for the integrating factor, 1 for finding an exact equation, 1 for the general solution, 1 for solving for y, 0.5 for finding C and 0.5 for choosing the negative root.]

Question 4. Version B. $3x^2 + xy + \left(\frac{x^3}{y} + x^2\right) y' = 0,\ y(1) = -3$

If $\tilde{M} = 3x^2 + xy$ and $\tilde{N} = \frac{x^3}{y} + x^2$, then

$$\frac{\partial \tilde{M}}{\partial y} = x \qquad\qquad \frac{\partial \tilde{N}}{\partial x} = \frac{3x^2}{y} + 2x$$

This equation is not exact. However, we can use an integrating factor:

$$\begin{aligned}\frac{\frac{\partial \tilde{M}}{\partial y} - \frac{\partial \tilde{N}}{\partial x}}{\tilde{N}} = \frac{-x - \frac{3x^2}{y}}{-3x^2 - xy} &= \frac{1}{y}\\ \mu &= e^{\int y^{-1} dy}\\ &= e^{\ln y}\\ &= y\end{aligned}$$

We can thus rewrite the equation as $(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$.

If $M = 3x^2y + xy^2$ and $N = x^3 + x^2y$, then

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy \qquad\qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

Hence the equation is exact. Let

$$\begin{aligned}F(x,y) &= \int M(x,y)dx + g(y)\\ &= \int (3x^2y + xy^2)dx + g(y)\\ &= x^3y + \frac{1}{2}x^2y^2 + g(y)\\ \frac{\partial F}{\partial y} &= x^3 + x^2y + g'(y) = N = x^3 + x^2y\\ \therefore\ g'(y) &= 0\\ g(y) &= k\\ F(x,y) &= x^3y + \frac{1}{2}x^2y^2 = C \text{ (where the constant } k \text{ has been absorbed)}\end{aligned}$$

We can find the solution since we have a quadratic in y:

$$\begin{aligned}\frac{1}{2}x^2y^2 + x^3y - C &= 0\\ y &= \frac{-x^3 \pm \sqrt{x^6 + 2x^2C}}{x^2}\end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned}\frac{9}{2} - 3 - C &= 0\\ C &= \frac{3}{2}\end{aligned}$$

Only the negative root corresponds to this initial condition. Hence the solution is

$$y = \frac{-x^3 - \sqrt{x^6 + 3x^2}}{x^2}$$

[1 for the integrating factor, 1 for finding an exact equation, 1 for the general solution, 1 for solving for y, 0.5 for finding C and 0.5 for choosing the negative root.]

Question 4. Version C. $3x^2y^2 + xy^3 + (x^3y + x^2y^2)y' = 0$, $y(1) = -4$

If $\tilde{M} = 3x^2y^2 + xy^3$ and $\tilde{N} = x^3y + x^2y^2$, then

$$\frac{\partial \tilde{M}}{\partial y} = 6x^2y + 3xy^2 \qquad \frac{\partial \tilde{N}}{\partial x} = 3x^2y + 2xy^2$$

This equation is not exact. However, we can use an integrating factor:

$$\begin{aligned} \frac{\frac{\partial \tilde{M}}{\partial y} - \frac{\partial \tilde{N}}{\partial x}}{-\tilde{M}} &= \frac{3x^2y + xy^2}{-3x^2y^2 - xy^3} = -\frac{1}{y} \\ \mu &= e^{\int -y^{-1}dy} \\ &= e^{-\ln y} \\ &= \frac{1}{y} \end{aligned}$$

We can thus rewrite the equation as $(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$.

If $M = 3x^2y + xy^2$ and $N = x^3 + x^2y$, then

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy \qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

Hence the equation is exact. Let

$$\begin{aligned} F(x,y) &= \int M(x,y)dx + g(y) \\ &= \int (3x^2y + xy^2)dx + g(y) \\ &= x^3y + \frac{1}{2}x^2y^2 + g(y) \\ \frac{\partial F}{\partial y} &= x^3 + x^2y + g'(y) = N = x^3 + x^2y \\ \therefore\ g'(y) &= 0 \\ g(y) &= k \\ F(x,y) &= x^3y + \frac{1}{2}x^2y^2 = C \text{ (where the constant } k \text{ has been absorbed)} \end{aligned}$$

We can find the solution since we have a quadratic in y:

$$\begin{aligned} \frac{1}{2}x^2y^2 + x^3y - C = 0 \\ y = \frac{-x^3 \pm \sqrt{x^6 + 2x^2C}}{x^2} \end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned} 8 - 4 - C &= 0 \\ C &= 4 \end{aligned}$$

Only the negative root corresponds to this initial condition. Hence the solution is

$$y = \frac{-x^3 - \sqrt{x^6 + 8x^2}}{x^2}$$

[1 for the integrating factor, 1 for finding an exact equation, 1 for the general solution, 1 for solving for y, 0.5 for finding C and 0.5 for choosing the negative root.]

Question 5. Version A. $t^6\frac{dQ}{dt}+(2-3t^5)Q=0$, $Q(1)=1$

Dividing, we have

$$\frac{dQ}{dt}+(2t^{-6}-3t^{-1})Q=0$$

The integrating factor is

$$\begin{aligned} I &= e^{\int(2t^{-6}-3t^{-1})dt} \\ &= e^{-2t^{-5}/5-3\ln t} \\ &= \frac{1}{t^3}e^{-2t^{-5}/5} \end{aligned}$$

The differential equation is then

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{t^3}e^{-2t^{-5}/5}Q\right) &= 0 \\ \frac{1}{t^3}e^{-2t^{-5}/5}Q &= C \end{aligned}$$

Applying the initial condition, we have

$$e^{-2/5}=C$$

Hence we have

$$\frac{1}{t^3}e^{-2t^{-5}/5}Q=e^{-2/5}$$

and the solution is

$$Q=t^3e^{-2/5}e^{2t^{-5}/5}$$

[1 mark for dividing, 1 mark for the integrating factor, 1 mark for the general solution, 1 mark for finding C, 1 mark for the answer]

Question 5. Version B. $t^8 \frac{dQ}{dt} + (3 - 4t^7)Q = 0$, $Q(1) = 1$

Dividing, we have

$$\frac{dQ}{dt} + (3t^{-8} - 4t^{-1})Q = 0$$

The integrating factor is

$$\begin{aligned} I &= e^{\int (3t^{-8} - 4t^{-1})dt} \\ &= e^{-3t^{-7}/7 - 4\ln t} \\ &= \frac{1}{t^4} e^{-3t^{-7}/7} \end{aligned}$$

The differential equation is then

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{t^4} e^{-3t^{-7}/7} Q\right) &= 0 \\ \frac{1}{t^4} e^{-3t^{-7}/7} Q &= C \end{aligned}$$

Applying the initial condition, we have

$$e^{-3/7} = C$$

Hence we have

$$\frac{1}{t^4} e^{-3t^{-7}/7} Q = e^{-3/7}$$

and the solution is

$$Q = t^4 e^{-3/7} e^{3t^{-7}/7}$$

[1 mark for dividing, 1 mark for the integrating factor, 1 mark for the general solution, 1 mark for finding C, 1 mark for the answer]

Question 5. Version C. $t^7\frac{dQ}{dt}+(4-5t^6)Q=0$, $Q(1)=1$

Dividing, we have

$$\frac{dQ}{dt}+(4t^{-7}-5t^{-1})Q=0$$

The integrating factor is

$$\begin{aligned} I &= e^{\int(4t^{-7}-5t^{-1})dt} \\ &= e^{-2t^{-6}/3-5\ln t} \\ &= \frac{1}{t^5}e^{-2t^{-6}/3} \end{aligned}$$

The differential equation is then

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{t^5}e^{-2t^{-6}/3}Q\right) &= 0 \\ \frac{1}{t^5}e^{-2t^{-6}/3}Q &= C \end{aligned}$$

Applying the initial condition, we have

$$e^{-2/3}=C$$

Hence we have

$$\frac{1}{t^5}e^{-2t^{-6}/3}Q=e^{-2/3}$$

and the solution is

$$Q=t^5e^{-2/3}e^{2t^{-6}/3}$$

[1 mark for dividing, 1 mark for the integrating factor, 1 mark for the general solution, 1 mark for finding C, 1 mark for the answer]

Question 6. Version A. $4y' + y = \cosh 8t$, $y(0) = 0$.

Taking the Laplace transform, we have

$$\begin{aligned} 4s\mathcal{L}[y] - y(0) + \mathcal{L}[y] &= \mathcal{L}[\cosh 8t] \\ (4s+1)\mathcal{L}[y] &= \frac{s}{s^2-64} \\ \mathcal{L}[y] &= \frac{s}{(s^2-64)(4s+1)} \end{aligned}$$

Taking the inverse, we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{s}{s^2-64}\right] * \mathcal{L}^{-1}\left[\frac{1}{4s+1}\right] \\ &= \cosh 8t * \frac{1}{4}e^{-t/4} \\ &= \int_0^t \cosh[8(t-w)]\frac{1}{4}e^{-w/4}dw \end{aligned}$$

Using integration by parts (or you can use the definition of cosh and sinh as below), we have

$$\begin{aligned} u &= \cosh[8(t-w)] & v' &= \frac{1}{4}e^{-w/4} \\ u' &= -8\sinh[8(t-w)] & v &= -\frac{1}{16}e^{-w/4} \end{aligned}$$

Then we have

$$y = -\frac{1}{16}\cosh[8(t-w)]e^{-w/4}\Big|_0^t - \frac{1}{2}\int_0^t \sinh[8(t-w)]e^{-w/4}dw$$

Using integration by parts again, we have

$$\begin{aligned} u &= \sinh[8(t-w)] & v' &= -\frac{1}{2}e^{-w/4} \\ u' &= -8\cosh[8(t-w)] & v &= -\frac{1}{8}e^{-w/4} \end{aligned}$$

Hence

$$\begin{aligned} y &= -\frac{1}{16}\cosh[8(t-w)]e^{-w/4}\Big|_0^t + \frac{1}{8}\sinh[8(t-w)]e^{-w/4}\Big|_0^t - \int_0^t \cosh[8(t-w)]e^{-w/4}dw \\ y &= -\frac{1}{16}\cosh[8(t-w)]e^{-w/4}\Big|_0^t + \frac{1}{8}\sinh[8(t-w)]e^{-w/4}\Big|_0^t - y \\ 2y &= -\frac{1}{16}\cosh[8(t-w)]e^{-w/4}\Big|_0^t + \frac{1}{8}\sinh[8(t-w)]e^{-w/4}\Big|_0^t \\ &= -\frac{1}{16}e^{-t/4} + \frac{1}{16}\cosh 8t - \frac{1}{8}\sinh 8t \\ y &= -\frac{1}{32}e^{-t/4} + \frac{1}{32}\cosh 8t - \frac{1}{16}\sinh 8t \\ &= -\frac{1}{32}e^{-t/4} - \frac{1}{64}e^{8t} + \frac{3}{64}e^{-8t} \end{aligned}$$

[1 mark for taking the Laplace transform, 2 marks for the convolution, 2 marks for the solution. If convolution is not used, no higher mark than 2.5 can be given.]

Question 6. Version B. $4y' + y = \cosh 6t$, $y(0) = 0$.

Taking the Laplace transform, we have

$$\begin{aligned} 4s\mathcal{L}[y] - y(0) + \mathcal{L}[y] &= \mathcal{L}[\cosh 6t] \\ (4s+1)\mathcal{L}[y] &= \frac{s}{s^2-36} \\ \mathcal{L}[y] &= \frac{s}{(s^2-36)(4s+1)} \end{aligned}$$

Taking the inverse, we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{s}{s^2-36}\right] * \mathcal{L}^{-1}\left[\frac{1}{4s+1}\right] \\ &= \cosh 6t * \frac{1}{4}e^{-t/4} \\ &= \int_0^t \cosh[6(t-w)]\frac{1}{4}e^{-w/4}dw \end{aligned}$$

Using the definition of cosh and sinh (or you can use integration by parts, as above), we have

$$\begin{aligned} \int_0^t \cosh[6(t-w)]\frac{1}{4}e^{-w/4}dw &= \frac{1}{8}\int_0^t \left(e^{6(t-w)} + e^{-6(t-w)}\right)e^{-w/4}dw \\ &= \frac{1}{8}\int_0^t \left(e^{6t}e^{-25w/4} + e^{-6t}e^{23w/4}\right)dw \\ &= \frac{1}{8}\left[-\frac{4}{25}e^{6t}e^{-25w/4} + \frac{4}{23}e^{-6t}e^{23w/4}\right]_0^t \\ &= \frac{1}{8}\left[\left(-\frac{4}{25}e^{6t}e^{-25t/4} + \frac{4}{23}e^{-6t}e^{23t/4}\right) - \left(-\frac{4}{25}e^{6t} + \frac{4}{23}e^{-6t}\right)\right] \\ &= \frac{1}{8}\left(-\frac{4}{25}e^{-t/4} + \frac{4}{23}e^{-t/4} + \frac{4}{25}e^{6t} - \frac{4}{23}e^{-6t}\right) \\ &= \frac{1}{575}e^{-t/4} + \frac{1}{50}e^{6t} - \frac{1}{46}e^{-6t} \end{aligned}$$

[1 mark for taking the Laplace transform, 2 marks for the convolution, 2 marks for the solution. If convolution is not used, no higher mark than 2.5 can be given.]

Question 6. Version C. $4y' + y = \cosh 3t$, $y(0) = 0$.

Taking the Laplace transform, we have

$$\begin{aligned} 4s\mathcal{L}[y] - y(0) + \mathcal{L}[y] &= \mathcal{L}[\cosh 3t] \\ (4s+1)\mathcal{L}[y] &= \frac{s}{s^2-9} \\ \mathcal{L}[y] &= \frac{s}{(s^2-9)(4s+1)} \end{aligned}$$

Taking the inverse, we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left[\frac{s}{s^2-9}\right] * \mathcal{L}^{-1}\left[\frac{1}{4s+1}\right] \\ &= \cosh 3t * \frac{1}{4}e^{-t/4} \\ &= \int_0^t \cosh[3(t-w)]\frac{1}{4}e^{-w/4}dw \end{aligned}$$

Using the definition of cosh and sinh (or you can use integration by parts, as above), we have

$$\begin{aligned} \int_0^t \cosh[3(t-w)]\frac{1}{4}e^{-w/4}dw &= \frac{1}{8}\int_0^t \left(e^{3(t-w)} + e^{-3(t-w)}\right)e^{-w/4}dw \\ &= \frac{1}{8}\int_0^t \left(e^{3t}e^{-13w/4} + e^{-3t}e^{11w/4}\right)dw \\ &= \frac{1}{8}\left[-\frac{4}{13}e^{3t}e^{-13w/4} + \frac{4}{11}e^{-3t}e^{11w/4}\right]_0^t \\ &= \frac{1}{8}\left[\left(-\frac{4}{13}e^{3t}e^{-13t/4} + \frac{4}{11}e^{-3t}e^{11t/4}\right) - \left(-\frac{4}{13}e^{3t} + \frac{4}{11}e^{-3t}\right)\right] \\ &= \frac{1}{8}\left(-\frac{4}{13}e^{-t/4} + \frac{4}{11}e^{-t/4} + \frac{4}{13}e^{3t} - \frac{4}{11}e^{-3t}\right) \\ &= \frac{1}{143}e^{-t/4} + \frac{1}{26}e^{6t} - \frac{1}{22}e^{-6t} \end{aligned}$$

[1 mark for taking the Laplace transform, 2 marks for the convolution, 2 marks for the solution. If convolution is not used, no higher mark than 2.5 can be given.]

Question 7.

x	8.3	8.6	8.7
y	17.56492	18.50515	18.82091

a) The second order Lagrange polynomial is

$$p_2(x) = \frac{(x-8.6)(x-8.7)}{(8.3-8.6)(8.3-9.8)}17.56492 + \frac{(x-8.3)(x-8.7)}{(8.6-8.3)(8.6-8.7)}18.50515 + \frac{(x-8.3)(x-8.6)}{(8.3-8.6)(8.3-8.7)}18.82091$$

[3]

b) We have

$$\begin{aligned} p_2(8.4) &= 0.5(17.56492) + 1(18.50515) - 0.5(18.82091) \\ &= 17.877155 \end{aligned}$$

[0.5]

c) The error satisfies

$$\begin{aligned} \epsilon_2(x) &= (x-8.3)(x-8.6)(x-8.7)\frac{f'''(t)}{3!} \\ \epsilon_2(8.4) &= (8.4-8.3)(8.4-8.6)(8.4-8.7)\frac{f'''(t)}{6} \\ &= 0.001 f'''(t) \end{aligned}$$

[1]

Using the bounds, we thus have

$$\begin{aligned} 0.001(-3.5) \leq \epsilon_2(8.4) &\leq 0.001(-0.8) \\ -0.0035 \leq \epsilon_2(8.4) &\leq -0.0008 \\ 17.877155 - 0.0035 \leq f(8.4) &\leq 17.877155 - 0.0008 \\ 17.873655 \leq f(8.4) &\leq 17.876355 \end{aligned}$$

[2]

[0.5] Thus the solution is accurate to one decimal place.