University of Ottawa
MAT 2384 Midterm Exam
Oct 28, 2014; Duration: 80 Minutes. Instructor: Robert Smith?

Family Name:______________________

First Name:______________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Good Luck!

Student number:____________________, Total marks: ___________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

Question 1. [5 points] Solve the initial-value problem

$$y\cos(x+y)dx + [3\sin(x+y) + y\cos(x+y)]dy = 0, \qquad y(0) = \frac{\pi}{2}$$

This is from the suggested problems #2, Q11.

Question 2. [4 points] Solve the initial-value problem

$$y + x^2 y' = e, \qquad y(1) = 0$$

Writing in standard form, we have

$$y' - \frac{1}{x^2} y = \frac{e}{x^2}$$

Hence an integrating factor is

$$I = e^{\int \frac{1}{x^2} dx} = e^{-x^{-1}}$$

Thus the equation is

$$\begin{aligned} e^{-x^{-1}} y' - \frac{1}{x^2} e^{-x^{-1}} y &= e \frac{e^{-x^{-1}}}{x^2} \\ \frac{d}{dx}\left(e^{-x^{-1}} y\right) &= e \frac{e^{-x^{-1}}}{x^2} \\ e^{-x^{-1}} y &= e \int \frac{e^{-x^{-1}}}{x^2} dx \end{aligned}$$

$$\begin{aligned} u &= -x^{-1} \\ \frac{du}{dx} &= \frac{1}{x^2} \\ dx &= x^2 du \end{aligned}$$

$$\begin{aligned} e^{-x^{-1}} y &= e \int \frac{e^u}{x^2} x^2 du \\ &= e e^u + c \\ &= e e^{-x^{-1}} + c \\ y &= e + c e^{x^{-1}} \end{aligned}$$

Applying the initial condition, we have

$$\begin{aligned} y(1) = e + ce &= 0 \\ c &= -1 \\ \therefore \qquad y &= e - e^{x^{-1}} \end{aligned}$$

Question 3. [4 points] Solve the initial-value problem

$$x^2y'' + xy' + 25y = 0, \qquad y(1) = 6, y'(1) = -15$$

This is an Euler-Cauchy equation, so solutions have the form $y \approx x^r$. Substituting, we have

$$\begin{aligned} r(r-1) + r + 25 &= 0 \\ r^2 + 25 &= 0 \\ r &= \pm 5i \end{aligned}$$

We thus have

$$\begin{aligned} y &= Ax^{5i} + Bx^{-5i} \\ &= Ae^{\ln x^{5i}} + Be^{\ln x^{-5i}} \\ &= Ae^{5i \ln x} + Be^{-5i \ln x} \\ &= c_1 \cos(5 \ln x) + c_2 \sin(5 \ln x) \end{aligned}$$

Differentiating, we have

$$y' = -\frac{5}{x} c_1 \sin(5 \ln x) + \frac{5}{x} c_2 \cos(5 \ln x)$$

Using the initial conditions, we have

$$\begin{aligned} y(1) = c_1 &= 6 \\ y'(1) = 5c_2 &= -15 \\ c_2 &= -3 \end{aligned}$$

Thus the solution is

$$y = 6 \cos(5 \ln x) - 3 \sin(5 \ln x)$$

Question 4. [4 points] Find the general solution for each of the following ODEs:

a) $y'' + 10y' + 25y = 0$

b) $y'' - 49y = 28xe^{7x}$

a) The characteristic equation is

$$\begin{aligned} \lambda^2 + 10\lambda + 25 &= 0 \\ (\lambda+5)^2 &= 0 \\ \lambda &= -5, -5 \end{aligned}$$

Thus the general solution is

$$y = Ae^{-5x} + Bxe^{-5x}$$

b) The characteristic equation is $\lambda^2 - 49 = 0$ so $\lambda = \pm 7$. Thus the homogeneous solution is

$$y_h = c_1 e^{7x} + c_2 e^{-7x}$$

Using variation of parameters, we have

$$\begin{aligned} y &= v_1 e^{7x} + v_2 e^{-7x} \\ y' &= 7v_1 e^{7x} - 7v_2 e^{-7x} \qquad\qquad v_1' e^{7x} + v_2' e^{-7x} = 0 \ (*) \\ y'' &= 49v_1 e^{7x} + 49 v_2 e^{-7x} + 7v_1' e^{7x} - 7v_2' e^{-7x} \\ y'' - 49y &= 7v_1' e^{7x} - 7v_2' e^{-7x} = 28xe^{7x} \ (**) \end{aligned}$$

Substituting (*) into (**), we have

$$\begin{aligned} 7v_1' e^{7x} + 7v_1' e^{7x} &= 28xe^{7x} \\ v_1' &= 2x \qquad\qquad v_2' = -2xe^{14x} \\ v_1 &= x^2 \qquad\qquad v_2 = -\frac{x}{7}e^{14x} + \frac{1}{7}\int e^{14x}dx \\ & \qquad\qquad\qquad = -\frac{x}{7}e^{14x} + \frac{1}{98}e^{14x} \end{aligned}$$

Thus the particular solution is

$$y_p = x^2 e^{7x} - \frac{x}{7}e^{7x} + \frac{1}{98}e^{7x}$$

Hence the general solution is

$$y = c_1 e^{7x} + c_2 e^{-7x} + x^2 e^{7x} - \frac{x}{7}e^{7x}$$

(where we have absorbed the final term of the particular solution into the arbitrary constants).

Question 5. [5 points] Find the general solution for the following ODE

$$y''' - y'' - 16y' - 20 = 0$$

The characteristic equation is

$$f(\lambda) = \lambda^3 - \lambda^2 - 16\lambda - 20$$

Testing a few values, we find

$$f(-2) = -8 - 4 + 32 - 20 = 0$$

Using long division, we discover

$$\begin{aligned} f(\lambda) &= (\lambda + 2)(\lambda^2 - 3\lambda - 10) \\ &= (\lambda + 2)(\lambda - 5)(\lambda + 2) \\ \lambda &= -2, -2, 5 \end{aligned}$$

Thus the general solution is

$$y = Ae^{-2x} + Bxe^{-2x} + Ce^{5x}$$

Question 6. [7 points] Consider the following set of data:

x	0.1	0.2	0.4
y	1.005	1.02	1.081

a) Write down the Lagrange polynomial of degree 2. Do not simplify.
b) Use your Lagrange polynomial to estimate the y value when $x = 0.3$.
c) To how many decimal places is this accurate? (Hint: $0.1 \leq f''' \leq 0.4$.)

a) The second order Lagrange polynomial is

$$p_2(x) = \frac{(x-0.2)(x-0.4)}{(0.1-0.2)(0.1-0.4)}1.005 + \frac{(x-0.1)(x-0.4)}{(0.2-0.1)(0.2-0.4)}1.02 + \frac{(x-0.1)(x-0.2)}{(0.4-0.1)(0.4-0.2)}1.081$$

b) We have

$$\begin{aligned} p_2(0.3) &= -0.335 + 1.02 + 0.3603333333 \\ &= 1.0453333333 \end{aligned}$$

c) The error satisfies

$$\begin{aligned} \epsilon_2(x) &= (x-0.1)(x-0.2)(x-0.4)\frac{f'''(t)}{3!} \\ \epsilon_2(0.3) &= (0.3-0.1)(0.3-0.2)(0.3-0.4)\frac{f'''(t)}{6} \\ &= -0.0003333333 f'''(t) \end{aligned}$$

Using the bounds, we thus have

$$\begin{aligned} -0.0003333333(0.4) \leq \epsilon_2(0.3) &\leq -0.0003333333(0.1) \\ -0.00013333333 \leq \epsilon_2(0.3) &\leq -0.00003333333 \\ \therefore -0.00013333333 + 1.0453333333 \leq f(0.3) &\leq -0.00003333333 + 1.0453333333 \\ 1.0452 \leq f(0.3) &\leq 1.0453 \end{aligned}$$

Thus the solution is accurate to <u>three</u> decimal places.

Formulas

$$
\begin{aligned}
p_1(x) &= f_0 + (x - x_0)f[x_0, x_1] \\
p_2(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
p_3(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&\vdots \\
f[x_j, x_{j+1}, \ldots, x_k] &= \frac{f[x_{j+1}, \ldots, x_k] - f[x_j, x_{j+1}, \ldots, x_{k-1}]}{x_k - x_j} \\
\epsilon_n(x) &= f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}
\end{aligned}
$$

$$
\begin{aligned}
p_1(x) &= L_0 f_0 + L_1(x) f_1 \\
p_2(x) &= L_0 f_0 + L_1(x) f_1 + L_2(x) f_2 \\
p_3(x) &= L_0 f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3 \\
&\vdots \\
L_i(x) &= \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \\
\epsilon_n(x) &= f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}
\end{aligned}
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