University of Ottawa
MAT 2384 Midterm Exam

Mar 1, 2017; Duration: 80 Minutes.
Instructor: Robert Smith?

Family Name:______________________ First Name:________________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to immediately leave the exam and academic fraud allegations will be filed, which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement:

  Signature:______________________
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed. All others will be confiscated.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- If you tear off any blank pages, they have to be handed in.
- Where it is possible to check your work, do so.
- Good luck!

Student number:____________________, Total marks: ___________

Problem	1	2	3	4	5	6	7
Marks							

You may complete any five of the first six questions, but you must answer Question 7. Indicate clearly which question you are not answering.

Question 1. [5 points] Explain some of the similarities and some of the differences between equations with constant coefficients and Euler–Cauchy equations. Give a second-order example of each. (Note: You must use full sentences. Only write on this page.)

(1 mark for at at least two similarities, 1 mark for at least two differences, 1 mark for each example, 1 mark for full sentences. The examples can come from elsewhere in the exam.)

Question 2. [5 points] Solve the initial-value problem

$$\left(\frac{1}{x^2} + y^2 + 3\right) dx + xy dy = 0, \qquad y(1) = -1$$

We have

$$\tilde{M} = \frac{1}{x^2} + y^2 + 3 \qquad\qquad \tilde{N} = xy$$

$$\frac{\partial \tilde{M}}{\partial y} = 2y \qquad\qquad \frac{\partial \tilde{N}}{\partial x} = y$$

The equation is not exact. However, we can find an integrating factor.

$$\frac{\frac{\partial \tilde{M}}{\partial y} - \frac{\partial \tilde{N}}{\partial x}}{\tilde{N}} = \frac{y}{xy} = \frac{1}{x}$$

$$\mu(x) = e^{\int 1/x dx} = e^{\ln x} = x$$

So the equation becomes $\left(\frac{1}{x} + xy^2 + 3x\right) dx + x^2 y dy = 0$.

$$M = \frac{1}{x} + xy^2 + 3x \qquad\qquad N = x^2 y$$

$$\frac{\partial M}{\partial y} = 2xy \qquad\qquad \frac{\partial N}{\partial x} = 2xy$$

This equation is exact. [1]

Hence we have

$$\begin{aligned} F &= \int \left(\frac{1}{x} + xy^2 + 3x\right) dx + g(y) \\ &= \ln|x| + \frac{x^2y^2}{2} + \frac{3x^2}{2} + g(y) \\ \frac{\partial F}{\partial y} &= x^2 y + g'(y) = N \end{aligned}$$

The general solution is thus

$$\ln|x| + \frac{x^2y^2}{2} + \frac{3x^2}{2} = C$$

[1]

Initial conditions: when $x = 1$, $y = -1$, so we have

$$\frac{1}{2} + \frac{3}{2} = C \qquad \Rightarrow \qquad C = 2$$

[1]

Solve for y:

$$y^2 = -\frac{2\ln|x|}{x^2} - 3 + \frac{4}{x^2}$$

$$y = -\sqrt{-\frac{2\ln|x|}{x^2} - 3 + \frac{4}{x^2}}$$

[1]

[Note: we choose the negative root because of the initial condition.] [1]

Question 3. [5 points] Solve the initial-value problem

$$t^5 \frac{dQ}{dt} + (4 - 5t^4)Q = t^5, \qquad Q(1) = 1$$

Rewrite as

$$\frac{dQ}{dt} + (4t^{-5} - 5t^{-1})Q = 1$$

This is a linear differential equation, so the integrating factor is

$$\begin{aligned} I &= e^{\int (4t^{-5} - 5t^{-1})dt} \\ &= e^{-t^{-4} - 5\ln t} \\ &= t^{-5}e^{-t^{-4}} \end{aligned}$$

[1]

Hence we have

$$\begin{aligned} \frac{d}{dt}\left(t^{-5}e^{-t^{-4}}Q\right) &= t^{-5}e^{-t^{-4}} \\ t^{-5}e^{-t^{-4}}Q &= \int t^{-5}e^{-t^{-4}}dt && u = -t^{-4} \\ &= \int \frac{1}{4}e^u du && \frac{du}{dt} = 4t^{-5} \\ &= \frac{1}{4}e^{-t^{-4}} + c && dt = \frac{1}{4}t^5 du \\ Q &= \frac{1}{4}t^5 + ct^5e^{t^{-41}} \end{aligned}$$

[2]

Using the initial condition, we have

$$\begin{aligned} Q(1) = \frac{1}{4} + ce &= 1 \\ c &= \frac{3}{4}e^{-1} \end{aligned}$$

[1]

Hence the solution is

$$Q = \frac{1}{4}t^4 + \frac{3}{4}e^{-1}t^5e^{t^{-4}}$$

[1]

Question 4. [5 points] Solve the initial-value problem

$$9x^2y'' + 3xy' + y = 0, \qquad y(1) = -1, y'(1) = 4$$

This is an Euler–Cauchy equation, so the characteristic equation is

$$\begin{aligned} 9r(r-1) + 3r + 1 &= 0 \\ 9r^2 - 6r + 1 &= 0 \\ (3r-1)(3r-1) &= 0 \\ r &= \frac{1}{3}, \frac{1}{3} \end{aligned}$$

[1]

The solution is thus in the form

$$y = At^{1/3} + Bt^{1/3} \ln t$$

[1]

Differentiating, we have

$$y' = \frac{1}{3}At^{-2/3} + Bt^{-2/3} + \frac{1}{3}Bt^{-2/3} \ln t$$

[1]

Applying initial conditions, we have

$$\begin{aligned} y(1) &= A = -1 \\ y'(1) &= \frac{1}{3}A + B = 4 \\ B &= \frac{13}{3} \end{aligned}$$

[1]

Hence the solution is

$$y = -t^{1/3} + \frac{13}{3}t^{1/3} \ln t$$

[1]

Question 5. [5 points] Find the general solution for each of the following ODEs:

$$\text{a) } y'' + 16y' + 64y = 0$$

$$\text{b) } y'' - 81y = 72xe^{9x}$$

a) The characteristic equation is $m^2 + 16m + 64 = 0$ or $(m+8)^2 = 0$ with $m = -8, -8$. Hence the general solution is

$$y = Ae^{-8x} + Bxe^{-8x}$$

[1]

b) The characteristic equation is $m^2 - 81 = 0$, so $m = \pm 9$. Hence

$$y_h = Ae^{9x} + Be^{-9x}$$

[0.5]

Using variation of parameters, we have

$$\begin{aligned} y_p &= u_1 e^{9x} + u_2 e^{-9x} \\ y_p' &= 9u_1 e^{9x} - 9u_2 e^{-9x} & u_1' e^{9x} + u_2' e^{-9x} &= 0 \\ y_p'' &= 81 u_1 e^{9x} + 81 u_2 e^{-9x} + 9u_1' e^{9x} - 9u_2' e^{-9x} & u_2' &= -u_1' e^{18x} \end{aligned}$$

Substituting in the original equation, we have

$$\begin{aligned} \cancel{81u_1e^{9x}} + \cancel{81u_2e^{-9x}} + 9u_1'e^{9x} - 9u_2'e^{-9x} - \cancel{81u_1e^{9x}} - \cancel{81u_2e^{-9x}} &= 72xe^{9x} \\ 9u_1'e^{9x} - 9u_2'e^{-9x} &= 72xe^{9x} \\ 9u_1'e^{9x} + 9u_1'e^{9x} &= 72xe^{9x} \\ 18u_1' &= 72x \\ u_1' &= 4x \qquad \Rightarrow \qquad u_1 = 2x^2 \\ u_2' &= -4xe^{18x} \\ w &= -4x \quad v' = e^{18x} \\ w' &= -4 \quad v = \frac{1}{18}e^{18x} \\ u_2 &= -\frac{2}{9}xe^{18x} + \frac{2}{9}\int e^{18x}dx \\ &= -\frac{2}{9}xe^{18x} + \frac{1}{81}e^{18x} \end{aligned}$$

[2]

Hence the particular solution is

$$\begin{aligned} y_p &= 2x^2e^{9x} + \left(-\frac{2}{9}xe^{18x} + \frac{1}{81}e^{18x}\right)e^{-9x} \\ &= 2x^2e^{9x} - \frac{2}{9}xe^{9x} + \frac{1}{81}e^{9x} \end{aligned}$$

[0.5]

It follows that the general solution is

$$\begin{aligned} y &= Ae^{9x} + Be^{-9x} + 2x^2e^{9x} - \frac{2}{9}xe^{9x} + \frac{1}{81}e^{9x} \\ &= \tilde{A}e^{9x} + Be^{-9x} + 2x^2e^{9x} - \frac{2}{9}xe^{9x} \end{aligned}$$

[1]

Question 6. [5 points] Solve the system of equations

$$x' = x + y$$
$$y' = -4x + y$$

with the initial conditions $x(0) = 8$, $y(0) = -5$.

We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -4 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = 1 \pm 2i$$

[1]

For $\lambda = 1 + 2i$, we have

$$\left[\begin{array}{cc|c} -2i & -1 & 0 \\ -4 & -2i & 0 \end{array}\right]$$

so the eigenvectors have equation

$$-2iv_1 + v_2 = 0$$

[1]

Letting $v_1 = s$, we have $v_2 = 2is$, so the eigenvectors are $\begin{bmatrix} 1 \\ 2i \end{bmatrix} s, s \in \mathbb{C}, s \neq 0$. Solutions are of the form

$$\begin{aligned} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} i \right) e^t(\cos(2t) + i\sin(2t)) \\ &= e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(2t) \right) + ie^t \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) \right) \end{aligned}$$

Taking the real and imaginary parts, we thus have the general solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(2t) \right) + c_2 e^t \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) \right)$$

[1.5]

Using the initial conditions, we have

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

[0.5]

The solution is thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = 8e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(2t) \right) - \frac{5}{2} e^t \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) \right)$$

[1]

Question 7. [7 points] Consider the following set of data:

x	1.1	1.2	1.4
y	2.731	2.767	2.897

a) Write down the Lagrange polynomial of degree 2. Do not simplify.
b) Use your Lagrange polynomial to estimate the y value when $x = 1.3$.
c) To how many decimal places is this accurate? (Hint: $-3.5 \leq f''' \leq -0.8$.)
a) The second order Lagrange polynomial is

$$p_2(x) = \frac{(x-1.2)(x-1.4)}{(1.1-1.2)(1.1-1.4)}2.731 + \frac{(x-1.1)(x-1.4)}{(1.2-1.1)(1.2-1.4)}2.767 + \frac{(x-1.1)(x-1.2)}{(1.4-1.1)(1.4-1.2)}2.897$$

[3]

b) We have

$$\begin{aligned} p_2(1.3) &= -0.9103333 + 2.767 + 0.96566667 \\ &= 2.82233333 \end{aligned}$$

[0.5]

c) The error satisfies

$$\begin{aligned} \epsilon_2(x) &= (x-1.1)(x-1.2)(x-1.4)\frac{f'''(t)}{3!} \\ \epsilon_2(1.3) &= (1.3-1.1)(1.3-1.2)(1.3-1.4)\frac{f'''(t)}{6} \\ &= -0.0003333333 f'''(t) \end{aligned}$$

[1]

Using the bounds, we thus have

$$\begin{aligned} -0.0003333333(-0.8) \leq \epsilon_2(1.3) &\leq -0.0003333333(-3.5) \\ 0.00026666667 \leq \epsilon_2(1.3) &\leq 0.00116666667 \\ 0.00026666667 + 2.82233333 \leq f(1.3) &\leq 0.00116666667 + 2.82233333 \\ 2.8226 \leq f(1.3) &\leq 2.8235 \end{aligned}$$

[2]

Thus the solution is accurate to <u>two</u> decimal places. [0.5]

Formulas

$$
\begin{aligned}
\mu(x) &= \exp\left[\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right] \\
\mu(y) &= \exp\left[\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right] \\
p_1(x) &= f_0 + (x - x_0)f[x_0, x_1] \\
p_2(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
p_3(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&\vdots \\
f[x_j, x_{j+1}, \ldots, x_k] &= \frac{f[x_{j+1}, \ldots, x_k] - f[x_j, x_{j+1}, \ldots, x_{k-1}]}{x_k - x_j} \\
\epsilon_n(x) &= f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}
\end{aligned}
$$

$$
\begin{aligned}
p_1(x) &= L_0 f_0 + L_1(x) f_1 \\
p_2(x) &= L_0 f_0 + L_1(x) f_1 + L_2(x) f_2 \\
p_3(x) &= L_0 f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3 \\
&\vdots \\
L_i(x) &= \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} \\
\epsilon_n(x) &= f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}
\end{aligned}
$$

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