Question	1-5	6-8	9	10	11	Total
Max	5	4	2	2	7	20
Marks						

[5pts]

Multiple-choice: please write your selection in the boxes provided here. You do not need to show any work. A correct answer is worth 1 point; an incorrect answer is worth 0 points.

Question	1	2	3	4	5
Answer	D	C	E	B	E

1. What is the slope of the line tangent to $x^3 + 2y^2x = 3$ at $(1, 1)$?

   A. 0

   B. $\frac{1}{4}$

   C. $\frac{1}{2}$

   D. $-\frac{5}{4}$

   E. -1

Diff. wrt x:

$3x^2 + 4yx\frac{dy}{dx} + 2y^2 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-2y^2 - 3x^2}{4xy}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2-3}{4} = \frac{-5}{4}$

2. Let $f(x) = 2x^3 + 4x^2 - 10$. What is the the absolute maximum and absolute minimum of $f(x)$ on $[-1, 2]$?

   A. max 22, min $-\frac{154}{9}$

   B. max -10, min $-\frac{154}{9}$

   C. max 22, min -10

   D. max 22, min -8

   E. There isn't an absolute maximum or minimum on this interval.

Find crit pts on $[-1,2]$:

$f'(x) = 6x^2 + 8x =: 0$

$\Rightarrow x(6x+8) = 0$

$\Rightarrow x = 0,\ x = \frac{-4}{3}$

but $\frac{-4}{3} \notin [-1,2]$

Compare:

$f(-1) = 2(-1)^3 + 4(-1)^2 - 10 = -2+4-10 = -8$

$f(0) = -10 \leftarrow$ abs min

$f(2) = 2(2)^3 + 4(2)^2 - 10 = 16+16-10 = 22 \leftarrow$ abs max

3. Let $f(x) = (x+7)^{\frac{2}{3}}$. Using a linear approximation of $f(x)$ near $a = 1$, what is an estimate for $9^{\frac{2}{3}}$?

   A. $\frac{9}{2}$

   B. $\frac{14}{3}$

   C. $\frac{17}{4}$

   D. $\frac{16}{3}$

   E. None of the above.

$f'(x) = \frac{2}{3}(x+7)^{-1/3}$

$f'(1) = \frac{2}{3} \cdot \frac{1}{\sqrt[3]{8}}$

$= \frac{1}{3}$

and $f(1) = 8^{2/3} = 4.$

$L(x) = f'(a)(x-a) + f(a)$

$= f'(1)(x-1) + f(1)$

$= \frac{1}{3}(x-1) + 4$

To estimate $9^{2/3}$, use $x=2$ in $L(x)$.

$L(2) = \frac{1}{3}(1) + 4 = \frac{13}{3}$

4. What is $\lim_{x \to 0} \frac{x \sin x}{\cos x - 1}$? $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\sin x + x \cos x}{-\sin x}$ (L'Hôpital's) $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\cos x + \cos x - x \sin x}{-\cos x}$ (L'Hôpital's)

$= \frac{2}{-1}$

A. 0

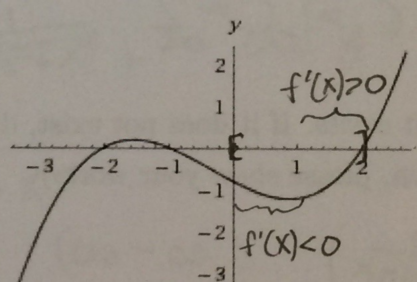
B. -2

C. 1

D. $-\infty$

E. ∞

5. Consider the graph of $y = f(x)$ as shown:



Exactly one of the following statements is true. Which statement is true?

A. $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [0, 2]$.

B. $f'(x) > 0$ and $f''(x) < 0$ for all $x \in [0, 2]$.

C. $f'(x) < 0$ and $f''(x) > 0$ for all $x \in [0, 2]$.

D. $f'(x) < 0$ and $f''(x) < 0$ for all $x \in [0, 2]$.

E. None of the above statements are true.

Short-answer questions: In questions 6-7, you do not need to show any steps or simplify your answers; a correct answer is worth full marks.

[1pts] 6. Find the general antiderivative of $f(x) = \sec^2 x + x^{\frac{3}{2}}$.

$$F(x) = \tan x + \frac{x^{5/2}}{(5/2)} + C$$

[1pts] 7. Find $f(x)$ if $f'(x) = \frac{1-x^2}{x}$ and $f(1) = \frac{3}{2}$.

$$f'(x) = \frac{1}{x} - \frac{x^2}{x} = \frac{1}{x} - x$$

So $f(x) = \ln|x| - \frac{x^2}{2} + 2$.

Then $f(x) = \ln|x| - \frac{x^2}{2} + C$

$$\frac{3}{2} = f(1) = \ln|1| - \frac{1^2}{2} + C$$

$$\Rightarrow \frac{3}{2} = 0 - \frac{1}{2} + C \Rightarrow C = 2.$$

[2pts] 8. Find $\lim_{x\to 0^+} (\frac{1}{x^2} - \frac{1}{\sin x})$, if it exists. If it does not exist, determine whether it is ∞, $-\infty$, or neither. (For this question, please show your work.)

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\sin x}\right) \quad (\infty - \infty)$$

$$= \lim_{x\to 0^+} \left(\frac{\sin x - x^2}{x^2 \sin x}\right) \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{\cos x - 2x}{2x\sin x + x^2\cos x}\right) \quad \text{(L'Hôpital's)}$$

When we let $x \to 0^+$ we get $\frac{a}{0}$ where $a \neq 0$, so limit is either ∞ or $-\infty$ (cannot be neither since this is a one-sided limit).

Check sign: when $x \to 0^+$: $\frac{+1}{(+)(+) + (+)\cdot 1} = (+)$

So $\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\sin x}\right) = \infty$.

Long-answer questions: in questions 9-10, you must show all relevant steps. A correct answer without any justification will not receive full marks.

[2pts]

9. Prove that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ using implicit differentiation.

Let $y = \arctan x$. We want $\frac{dy}{dx}$.

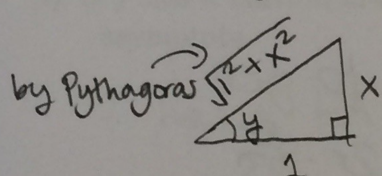
$\Rightarrow \tan y = x \quad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$

Diff. wrt x:

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \cos^2 y$$

We need y in terms of x. We know $\tan y = x$:



$\left(\frac{opp}{adj} = \frac{x}{1}\right)$

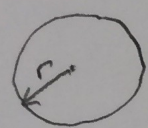
Then $\cos y = \frac{1}{\sqrt{1^2+x^2}}$, so $\cos^2 y = \frac{1}{1+x^2}$.

Hence $\frac{dy}{dx} = \frac{1}{1+x^2}$.

$\therefore \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

[2pts] 10. A spherical water balloon is filled in such a way that its surface area is always increasing at a rate of 20% of its current surface area per second. How quickly is its radius increasing when its surface area is π cm^2? *The surface area of a sphere is given by* $A = 4\pi r^2$. *Be sure to include the correct units in your final answer.*

Let r be the radius (cm) and S be the surface area (cm^2) of the balloon.



We are told: $\frac{dS}{dt} = 0.2S$

We want $\frac{dr}{dt}$, and by the chain rule we must write $\frac{dr}{dt} = \frac{dr}{dS} \cdot \frac{dS}{dt}$

To find $\frac{dr}{dS}$, we know: $S = 4\pi r^2$

Diff wrt S to get

$$1 = 4\pi \cdot 2r \cdot \frac{dr}{dS}$$

$$\Rightarrow \frac{dr}{dS} = \frac{1}{8\pi r}$$

Then $\frac{dr}{dt} = \frac{1}{8\pi r} \cdot 0.2S$

When $S = \pi$, we have $\pi = 4\pi r^2 \Rightarrow r^2 = \frac{1}{4}$

$$\Rightarrow r = \frac{1}{2}.$$

So $\left.\frac{dr}{dt}\right|_{\substack{S=\pi \\ r=\frac{1}{2}}} = \frac{1}{8\pi(\frac{1}{2})} \cdot 0.2\pi = \frac{0.2}{8 \cdot 0.5} = \frac{2}{8 \cdot 5} = \boxed{\frac{1}{20} \text{ cm/s}}$

Answer ought to be in cm/s; let's check:

$\frac{dr}{dS} = \frac{1}{8\pi r}$ (1/cm) So $\frac{dr}{dS} \cdot \frac{dS}{dt}$ is in $\frac{1}{cm} \cdot \frac{cm^2}{s} = \frac{cm}{s}$

$\frac{dS}{dt} = 0.2S$ (cm^2/s - not at all obvious, but we know $\frac{dS}{dt}$ should be $\frac{\text{area}}{\text{time}}$ in cm^2/s)

Sketching question: answer all preliminary questions, then sketch your graph on the next page. Your sketch will be evaluated based on how well it reflects the information you gather beforehand.

11. Let $f(x) = \frac{x^2-x-2}{x^2-9}$.

[1pts] a) Find all intercepts and vertical asymptotes of f.

x-int: $x^2-x-2=0$
$\Rightarrow (x-2)(x+1)=0$
$x=2, x=-1$

y-int: $\frac{0^2-0-2}{0^2-9}=\frac{2}{9}$

VA: $x=\pm 3$ (from $x^2-9=0$
$\Rightarrow (x+3)(x-3)=0$)

[1pts] b) If f has a horizontal or slant asymptote, give the equation of the horizontal or slant asymptote.

f has a HA because deg of num = deg of denom.

HA @ $y = \lim_{x\to\infty} \frac{\boxed{x^2} - x - 2}{\boxed{x^2} - 9}$

$= 1.$

(For your convenience: $f(x) = \frac{x^2-x-2}{x^2-9}$.)

[2pts] **c)** Give the intervals of increase and decrease of f, and give the x-values at which local maxima and local minima occur.

$$f'(x) = \frac{(2x-1)(x^2-9)-(x^2-x-2)(2x)}{(x^2-9)^2}$$

$$= \frac{2x^3-x^2-18x+9-2x^3+2x^2+4x}{(x^2-9)^2}$$

$$= \frac{x^2-14x+9}{(x^2-9)^2}$$

Quad Formula:

$$x = \frac{14 \pm \sqrt{14^2-9\cdot4}}{2}$$

$$= 7 \pm \tfrac{1}{2}\sqrt{14^2-36}$$

$$= 7 \pm \tfrac{1}{2}\sqrt{160}$$

$$= 7 \pm 2\sqrt{10}$$

$f'(x)$ DNE @ $x = \pm 3$ (VAs)

$f'(x) = 0$ when $x^2-14x+9=0$

$\sqrt{10} \approx 3 +$ a bit.

$$f'(x) = \frac{(x-7-\sqrt{10})(x-7+\sqrt{10})}{(x^2-9)^2}$$

	-3		$7-2\sqrt{10}$		3		$7+2\sqrt{10}$	
$\frac{(-)(-)}{(+)}$		$\frac{(-)(-)}{(+)}$		$\frac{(-)(+)}{(+)}$		$\frac{(-)(+)}{(+)}$		$\frac{(+)(+)}{(+)}$
$=(+)$		$=(+)$		$=(-)$		$=(-)$		$=(+)$

So f increasing on $(-\infty,-3)\cup(-3,7-2\sqrt{10})\cup(7+2\sqrt{10},\infty)$

f decreasing on $(7-2\sqrt{10},3)\cup(3,7+2\sqrt{10})$.

local max @ $x = 7-\sqrt{10}$

local min @ $x = 7+\sqrt{10}$

[2pts] **e)** Sketch your graph, making sure to label all asymptotes, intercepts, and the locations of any maxima, minima, and points of inflection (you do not need to find the y-values for such points). *You may detach this page, but please remember to get it stapled when you turn your test in.*