MATH2004 Notes - By Eric Hua

Contents

0.1. Fourier Series

Pre-knowledge

1. Trig Identities:

$$\sin(n\pi + \frac{\pi}{2}) = (-1)^n; \quad \cos(n\pi) = (-1)^n, \quad n \text{ is an integer.}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}.$$

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}, \quad \cos a \cos b = \frac{\cos(a+b) + \cos(a-b)}{2}.$$

$$\sin a \cos b = \frac{\sin(a+b) + \sin(a-b)}{2}.$$

2. A function $f(x)$ is called $p-periodic$ if $p > 0$ is the smallest number such that $f(x+p) = f(x)$ for any x. The number p is called the period. For example, $\cos kx$ and $\sin kx$ are $\frac{2\pi}{k}$-periodic.

3. Odd-Even function: If $f(-x) = -f(x)$ for all $x \in [-a, a]$, then $f(x)$ is odd on $[-a, a]$; If $f(-x) = f(x)$ for all $x \in [-a, a]$, then $f(x)$ is even on $[-a, a]$. For example, $\sin kx$ is odd, $\cos kx$ is even.

   - If $f(x)$ is odd on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 0$.
   - If $f(x)$ is even on $[-a, a]$, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

4. If m, n are non-negative integers, and $m \neq n$, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0.$$

   Also, even if we drop the restriction $m \neq n$,

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0.$$

5. The trigonometric system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx, ...$$

   is orthogonal on the interval $-\pi \leq x \leq \pi$.

6. If n is a positive integer, then

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi.$$

7. Integration by parts: $\int_a^b u dv = uv|_a^b - \int_a^b v du$.

8. Integration by substitution: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

0.2. Definition of Fourier Series.

Fourier series are named in honor of Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series. Fourier series have many applications such as, solving partial differential equations, signal precessing, image processing.

A function $f(x)$ is **piecewise continuous** in interval (a, b) if we have $a = t_0 < t_1 < \ldots < t_m = b$, such that f(x) is continuous in each interval (t_i, t_{i+1}) and the limits $\lim_{x \to t_i^-} f(x)$ and $\lim_{x \to t_i^+} f(x)$ exist for all $i = 0, 1, 2, \ldots, m$. In the following, we assume that both f and f' are piecewise continuous.

Definition. Let $f(x)$ be $2L$-periodic function. Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\}. \tag{1}$$

This series is called the (full) Fourier series for $f(x)$. The coefficients a_n $(n \geq 0)$ are called the Fourier cosine coefficients, and the coefficients b_n $(n \geq 1)$ are called the Fourier sine coefficients.

Remark. The "=" occurs at every $x \in [-L, L]$ where f is continuous. If we omit the condition where f is continuous at x, then we may write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})).$$

Theorem. The Fourier coefficients can be calculated as follows:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx \qquad n = 0, 1, 2, \ldots. \tag{2}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx \qquad n = 1, 2, 3, \ldots. \tag{3}$$

Proof. The coefficient a_0 is the simplest to find: integrating (1) from $-L$ to L,

$$\begin{aligned} \int_{-L}^{L} f(x)\, dx &= \int_{-L}^{L} \frac{a_0}{2}\, dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^{L} \cos(\frac{n\pi x}{L})\, dx + a_n \int_{-L}^{L} \sin(\frac{n\pi x}{L})\, dx \right\} \\ &= \int_{-L}^{L} \frac{a_0}{2}\, dx \end{aligned}$$

The series on the right vanishes, and we find that

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx.$$

We do the same thing to compute, say, b_m, except that first we multiply (1) through by $\sin(\frac{m\pi x}{L})$. We get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx &= \int_{-L}^{L} \frac{a_0}{2}\sin(\frac{m\pi x}{L})\,dx + \\ &\sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx + b_n\int_{-L}^{L}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})\,dx\right\}.\end{aligned}$$

What is important to notice is that *all* of the integrals on the right side vanish, except for the one multiplying b_m. The equation for b_m becomes

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi x}{L})\,dx \qquad m = 1, 2, 3, \ldots.$$

Likewise we can get the formula for a_m.

Formulas (2) and (3) allow us to compute the Fourier coefficients of f.

Remark 1. Even though f is defined only on $[-L, L]$, the right-hand side of (1) is $2L$-periodic, so we could view f as being defined over the whole line, but $2L$-periodic as well.

Remark 2. If f is even on $[-L, L]$, then $f(x)\sin(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $b_n = 0$ for all $n \geq 1$; and $f(x)\cos(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \qquad n = 0, 1, 2, \ldots.$$

Remark 3. If f is odd on $[-L, L]$, then $f(x)\cos(\frac{m\pi x}{L})$ is odd on $[-L, L]$, so $a_n = 0$ for all $n \geq 0$; and $f(x)\sin(\frac{m\pi x}{L})$ is even on $[-L, L]$, so

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \qquad n = 1, 1, 2, \ldots.$$

Remark 4. If f is 2π-periodic (i.e., $L = \pi$), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\},$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0, 1, 2, \ldots. \tag{5}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1, 2, 3, \ldots. \tag{6}$$

Example 1. *Let $f(x) = x$, for all $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic. Compute the Fourier coefficients.*

Notice that $\cos(nx)$ is an even function, while f and $\sin(nx)$ are odd functions.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx = \frac{2}{\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right) = (-1)^{n+1}\frac{2}{n}
\end{aligned}$$

Notice that a_0, a_n are 0 because x and $x\cos(nx)$ are odd functions. Hence the Fourier series for $f(x) = x$ is:

$$\begin{aligned}
x &= a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) = \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx), \quad \forall x \in (-\pi, \pi)
\end{aligned}$$

Example 2. *Let $f(x) = x^2$, $x \in [-\pi, \pi)$, and $f(x)$ be 2π-periodic. Compute the Fourier coefficients.*

Since f is *even* ($f(x) = f(-x)$ for all x), then $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2,$$

and for $n \geq 1$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx \\
&= \frac{1}{n\pi}\left\{x^2\sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(nx)\,dx\right\} \\
&= \frac{(-1)^n \cdot 4}{n^2}.
\end{aligned}$$

Thus for $x \in (-\pi, \pi)$,

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n \cdot 4}{n^2}\cos(nx).$$

Example 3. *Let*

$$f(x) = \begin{cases} 0, & \textit{for } x \in [-\pi, 0); \\ 1, & \textit{for } x \in (0, \pi). \end{cases},$$

and let $f(x)$ be 2π-periodic. Find the Fourier series of $f(x)$.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right) = 1, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cos(nx)dx + \int_{0}^{\pi} 1\cos(nx)dx\right) = \frac{1}{n\pi}\sin(nx)|_0^\pi = 0, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\sin(nx)dx + \int_{0}^{\pi} 1\sin(nx)dx\right) = -\frac{1}{n\pi}\cos(nx)|_0^\pi \\
&= \frac{1}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{2}{n\pi}, & \text{for odd } n; \\ 0, & \text{for even } n. \end{cases}
\end{aligned}$$

Hence the Fourier series for $f(x)$ is:

$$\begin{aligned}
f(x) &\sim \frac{1}{2} + \sum_{odd\ n} \frac{2}{n\pi}\sin(nx) = \\
&= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi}\sin(2n+1)x, \quad \forall x \in (-\pi, \pi).
\end{aligned}$$

Example 4. *Let $f(x)$ be 2-periodic and* $f(x) = \begin{cases} x, & 0 < x < 1; \\ 0, & -1 < x \le 0. \\ 0.5, & x = -1, 1. \end{cases}$

In this case, $L = 1$.

$$\begin{aligned}
a_0 &= \int_0^1 x\,dx = \frac{1}{2}, \\
a_n &= \int_0^1 x\cos(n\pi x)\,dx = \frac{(-1)^n - 1}{(n\pi)^2}, \\
b_n &= \int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^n}{n\pi}.
\end{aligned}$$

The full Fourier series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^n}{n\pi}\sin(n\pi x)\right].$$

Example 5. *(Periodic square wave) Find the Fourier series of the function*

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1; \\ k, & \text{if } -1 < x < 1; \\ 0, & \text{if } 1 < x < 2. \end{cases}$$

with period 4.

Solution: Since the period is 4, so $p = 4, L = 2$. From (**??**)-(**??**) we obtain

$$\begin{aligned} a_0 &= \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}, \\ a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi}{2}x\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & \text{if } n \text{ is even;} \\ (-1)^m \frac{2k}{n\pi}, & \text{if } n = 2m+1, \end{cases} \\ b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi}{2}x\,dx = 0, \quad (n = 1, 2, ...). \end{aligned}$$

Hence

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

0.3. Geometric interpretation of Fourier series.

If we let

$$S_1 = \frac{1}{2} + \frac{2}{\pi}\sin x,$$

$$S_3 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x,$$

$$S_5 = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x,$$

each partial sum is a continuous function that approximates the discontinuous function $f(x) = \begin{cases} 0, & \text{for } x \in [-\pi, 0); \\ 1, & \text{for } x \in (0, \pi). \end{cases}$, on the interval $(-\pi, \pi)$. The bigger n, the better the approximation.

0.4. Fourier cosine and Fourier sine series.

Half-range Expansions

Let $f(x)$ be define on $(0, L)$. Three special extensions are important:
(i) Consider $f(x)$ as an odd function on $(-L, L)$, i.e.,

$$f_{odd}(x) = \begin{cases} f(x), & x \in (0, L); \\ -f(-x), & x \in (-L, 0). \end{cases}$$

Then $f(x)$ is $2L-$periodic. Thus $a_n = 0$ for all n, (1) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{7}$$

which is called **Fourier sine series** of f, where

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots.$$

(ii) Consider $f(x)$ as an even function on $(-L, L)$, i.e.,

$$f_{even}(x) = \begin{cases} f(x), & x \in (0, L); \\ f(-x), & x \in (-L, 0). \end{cases}$$

Then $f(x)$ is $2L-$periodic. Thus $b_n = 0$ for all n, (1) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{8}$$

which is called **Fourier cosine series** of f, where

$$a_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\,dx \qquad m = 0, 1, 2, 3, \ldots.$$

The cosine and sine series here are known as HALF-RANGE EXPANSIONs.

(iii) Consider $f(x)$ as $L-$periodic. Then half-period is $L/2$.

$$a_n = \frac{1}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L/2})\,dx \qquad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L/2}\int_0^L f(x)\sin(\frac{n\pi x}{L/2})\,dx \qquad n = 1, 2, 3, \ldots.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)] \tag{9}$$

which is called **(full) Fourier series** of f.

Example 6. *Let* $f(x) = \begin{cases} x, & 0 \le x < 1. \\ 0, & 1 \le x < 2; \end{cases}$. *Find the Fourier sine series and Fourier cosine series.*

Solution: (i) Fourier sine series: for $m = 1, 2, 3, \ldots,$

$$\begin{aligned} b_m &= \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\sin(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\sin(\frac{m\pi x}{2})\,dx = \left[-\frac{2}{m\pi}x\cos(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi x}{2})\right]_0^1 \\ &= -\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2}). \end{aligned}$$

The Fourier sine series is

$$f(x) = \sum_{m=1}^{\infty}\left[-\frac{2}{m\pi}\cos(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\sin(\frac{m\pi}{2})\right]\sin(\frac{m\pi x}{2}).$$

We have 4-periodic **odd** extension of f:

$$\tilde{f}(x) = \begin{cases} 0, & -2 \le x < -1; \\ x, & -1 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+4) = \tilde{f}(x), & \text{otherwise.} \end{cases}$$

In this case, $L = 2$.

(ii) Fourier cosine series:for $m = 0, 1, 2, 3, \ldots,$

$$\begin{aligned} a_m &= \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi x}{L})\,dx = \frac{2}{2}\int_0^2 f(x)\cos(\frac{m\pi x}{2})\,dx \\ &= \int_0^1 x\cos(\frac{m\pi x}{2})\,dx = \left[\frac{2}{m\pi}x\sin(\frac{m\pi x}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi x}{2})\right]_0^1 \\ &= \frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}. \end{aligned}$$

The Fourier cosine series is

$$f(x) = \sum_{m=1}^{\infty}\left[\frac{2}{m\pi}\sin(\frac{m\pi}{2}) + \frac{4}{m^2\pi^2}\cos(\frac{m\pi}{2}) - \frac{4}{m^2\pi^2}\right]\cos(\frac{m\pi x}{2}).$$

We have 4-periodic **even** extension of f:

$$\tilde{f}(x) = \begin{cases} 0, & -2 \le x < -1; \\ -x, & -1 \le x < 0; \\ x, & 0 \le x < 1; \\ 0, & 1 \le x < 2. \\ \tilde{f}(x+4) = \tilde{f}(x), & \text{otherwise.} \end{cases}$$

In this case, $L = 2$.

0.5. Points of discontinuity and convergence.

In equation (1), "=" means that the series on the right converges to the function on the left at each point x. It often happens that the Fourier series of a function f fails to converge to that function, in particular at the points of discontinuity of f.

The facts are:

- **The Fourier Theorem**: If the function $f(x)$ is piecewise continuously differentiable then its Fourier series converges for every x to the average value

$$f_{av}(x) = \frac{f(x+) + f(x-)}{2}, \tag{10}$$

  where

$$f(x+) = \lim_{t \to x+} f(t), \quad f(x-) = \lim_{t \to x-} f(t).$$

- At the points where $f(x)$ is continuous, $f_{av}(x) = f(x)$.

All the functions we shall consider in the sequel are piecewise continuously differentiable, and therefore the Fourier series will represent the function. In order to ensure that the Fourier series of function $f(x)$ converges to that function at every $x \in \mathbb{R}$, sometimes it is necessary to redefine $f(x)$ at the points of discontinuity x, so that $f_{av}(x) = f(x)$.

Example 7. *Let*

$$f(x) = \begin{cases} -1, & \text{for } x \in (-\pi, 0); \\ x, & \text{for } x \in (0, \pi); \\ 0, & \text{for } x = 0, \pi, -\pi. \end{cases}$$

Determine the sums to which the series converges at $x = 0, \pm\pi, 88\pi, 101\pi$.

Solution: The sum=-0.5 at 0; $\frac{\pi - 1}{2}$ at $\pm\pi$.

0.6. Termwise differentiation and integration.

In applications, Fourier series can be as a solution to a differential equation. To find such a solution, we need to differentiate the series.

(i) If f is $2L$-periodic and continuous, with f' and f'' piecewise continuous, then

$$f'(x) = \sum_{n=1}^{\infty} \Big[-\frac{n\pi}{L} a_n \sin(\frac{n\pi x}{L}) + \frac{n\pi}{L} b_n \cos(\frac{n\pi x}{L}) \Big] \tag{11}$$

at every x where f' is continuous. At a point x of discontinuity of f', the series converges to $\frac{f'(x+)+f'(x-)}{2}$.

(ii) If f is piecewise continuous on $[-L, L]$, then for any $-L \leq c \leq x \leq L$,

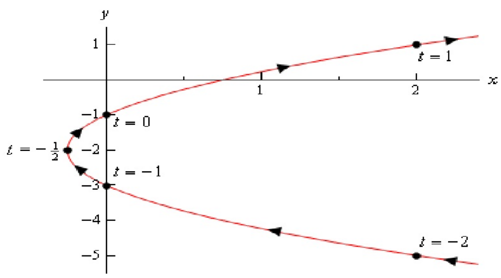
$$\int_c^x f(x)dx = \int_c^x \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \int_c^x \Big[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \Big] dx. \tag{12}$$

Chapter 10: Parametric Equations and Polar Coordinates

10.1. Curves Defined By Parametric Equations

Instead of defining y in terms of x, $y = f(x)$, we define both x and y in terms of a third variable called a parameter as follows: $x = f(t), y = g(t)$. This third variable t is called a parameter. The collection of points $(x, y) = (f(t), g(t))$ that we get by letting t be all possible values is the graph of the parametric equations and is called the parametric curve.

Example 8. *Sketch the parametric curve for the following set of parametric equations.* $x = t^2 + t, y = 2t - 1$.



Remark. If you eliminate the parameter t.

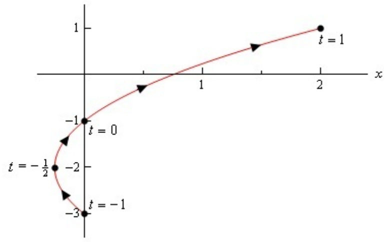
Now consider the following parametric curve with limits on the parameter:

$$x = f(t), y = g(t), a \leq t \leq b.$$

Then $(f(a), g(a))$ is called initial point, $(f(b), g(b))$ is called terminal point.

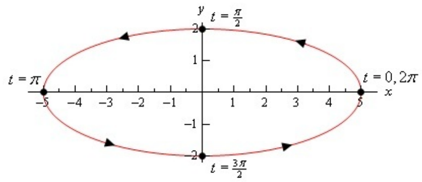
Example 9. *Sketch the parametric curve for the following set of parametric equations:*

$$x = t^2 + t, y = 2t - 1, -1 \leq t \leq 1.$$



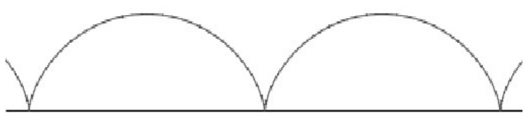
Example 10. *Sketch the parametric curve for the following set of parametric equations:*

$$x = 5\cos t, y = 2\sin t, 0 \leq t \leq 2\pi.$$



Example 11. *Sketch the Cycloid :*

$$x = r(t - \sin t), y = r(1 - \cos t), , -\infty < t < \infty.$$



10.2. Calculus with Parametric Equations

Tangents

We want to find the tangent lines to the parametric equations given by, $x = f(t), y = g(t)$. By Chain Rule, we have

- First Derivative for Parametric Equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \ \textit{provided} \ \frac{dx}{dt} \neq 0.$$

- Second Derivative for Parametric Equations:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \ \textit{provided} \ \frac{dx}{dt} \neq 0.$$

Example 12. *Find the tangent line(s) to the parametric curve given by*

$$x = t^5 - 4t^3, y = t^2, \ at \ (0, 4).$$

Solution. At first we need the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}.$$

When $(x, y) = (0, 4)$, $t = \pm 2$. At $t = -2$, the slope of the tangent line is: The tangent line at $t = -2$ is: $y = 4 - x/8$; At $t = 2$, the slope is: The tangent line (at $t = 2$) is: $y = 4 + x/8$.

Example 13. *Find the points where the following parametric equations will have horizontal or vertical tangents:*

$$x = t^3 - 3t, y = 3t^2 - 9.$$

Solution. Horizontal Tangents: $dy/dt = 0, 6t = 0, t = 0$. Therefore, the only horizontal tangent will occur at the point $(x, y) = (0, -9)$.

Vertical Tangents: $dx/dt = 0$ (dy/dx undefined). In this case we need to solve, $3(t^2 - 1) = 0, \Rightarrow t = 1, -1$. The two vertical tangents will occur at the points (2,-6) and (-2,-6).

Example 14. *Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down.*

$$x = 1 - t^2, y = t^7 + t^5.$$

Solution: To study concavity, we need the second derivative.

$$\frac{dy}{dx} = -\frac{7t^5 + 5t^3}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{35t^3 + 15t}{4}.$$

From $\frac{d^2y}{dx^2} = 0$ we imply that t=0. When $t < 0$, $\frac{d^2y}{dx^2} < 0$, the parametric curve will be concave down; when $t > 0$, $\frac{d^2y}{dx^2} > 0$, the parametric curve will be concave up.

Area

Here we will find a formula for determining the area under a parametric curve given by the parametric equations:

$$x = f(t), y = g(t), \alpha \le t \le \beta.$$

We assume that the curve is traced out exactly once as t increases from α to β. As we know, the area under the curve $y = F(x), a \le x \le b$ is

$$A = \int_a^b y(x)\, dx = \int_\alpha^\beta y(t)x'(t)dt, \;\; where \;\; x(\alpha) = a, x(\beta) = b.$$

Example 15. *Find the area under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t).$$

Solution: $y = 0 \Rightarrow t = n\pi$. Thus

$$A = \int_{\alpha}^{\beta} y(t)x'(t)dt = \int_{0}^{2\pi} r^2(1-\cos t)^2 dt = r^2 \int_{0}^{2\pi} (1 - 2\cos t + \frac{1+\cos 2t}{2})dt = 3\pi r^2.$$

Arc Length

Here we will find a formula for determining the arc length to a parametric curve given by the parametric equations:

$$x = f(t), y = g(t), \alpha \leq t \leq \beta.$$

We assume that the curve is traced out exactly once as t increases from α to β. Also, for the purposes of the derivation that we're going to use, we will assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$dx/dt \geq 0, \alpha \leq t \leq \beta.$$

The arc length formula is given by

$$L = \int_{a}^{b} \sqrt{1 + [y'(x)]^2}\, dx = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt, \;\; where \;\; x(\alpha) = a, x(\beta) = b.$$

Example 16. *Find the length under one arch of cycloid:*

$$x = r(t - \sin t), y = r(1 - \cos t), \;\; 0 \leq t \leq 2\pi.$$

Solution:

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_{0}^{2\pi} \sqrt{[r(1-\cos t)]^2 + [r\sin t]^2} dt$$

$$= r\int_{0}^{2\pi} \sqrt{2 - 2\cos t} dt = r\int_{0}^{2\pi} 2\left|\sin\frac{t}{2}\right| dt = 8r.$$

Surface Area

Here we will determine the surface area of a region obtained by rotating a parametric curve about the x-axis or y-axis. We will rotate the parametric curve given by

$$x = f(t), y = g(t), \alpha \leq t \leq \beta$$

about the x-axis or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β.

(1) Area of a surface obtained by rotating a curve about x-axis:

$$S = \int_a^b 2\pi y\, ds = \int_\alpha^\beta 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \ \ ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

(2) Area of a surface obtained by rotating a curve about y-axis: just need to change y to x:

$$S = \int_a^b 2\pi x\, ds = \int_\alpha^\beta 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \ \ ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

Example 17. *Find the surface area of one arch of cycloid rotating about x-axis:*

$$x = r(t - \sin t), y = r(1 - \cos t), \ 0 \le t \le 2\pi.$$

Solution:

$$S = \int_\alpha^\beta 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} 2\pi r(1-\cos t)\sqrt{[r(1-\cos t)]^2 + [r\sin t]^2}dt$$

$$= 2\pi r^2 \int_0^{2\pi} (1-\cos t)\sqrt{2 - 2\cos t}dt = 8\pi r^2 \int_0^{2\pi} \sin^3 \frac{t}{2}dt = 16\pi r^2 \int_0^{\pi} \sin^3 \phi d\phi$$

$$= -16\pi r^2 \int_0^{\pi} (1-\cos^2\phi) d\cos\phi = \frac{64}{3}\pi r^2.$$

Example 18. *Show that the surface area of a sphere of radius r is $4\pi r2$.*

Solution: The sphere is obtained by rotating the semicircle

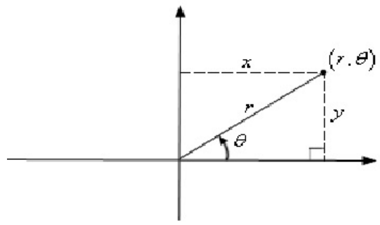
$$x = r\cos t, y = r\sin t, 0 \le t \le \pi$$

about the x-axis.

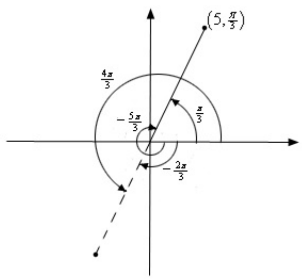
10.3. Polar Coordinates

Polar coordinates

We chose a point in the plane that is called the pole (or origin) and is labelled O. Then we draw a ray starting at O, along positive x-axis, which is called the polar axis. Let P be a point in the plane. Let r be the distance from P to O, let θ be the angle between OP and the polar axis. Then P can be represented by the ordered pair (r, θ). We call r and θ polar coordinates:



Example 19. *Sketch of several polar coordinates:*



Polar ⇄ Cartesian Conversion Formulas:

$$x = r\cos\theta, y = r\sin\theta; \quad r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}.$$

Example 20. *Convert each of the following points into the given coordinate system.*
(a) $(-4, \frac{2\pi}{3})$ *into Cartesian coordinates. (b) (-1,-1) into polar coordinates.*

Solution: (a) $(x, y) = (2, -2\sqrt{3})$.

(b) $r = \sqrt{x^2 + y^2} = \sqrt{2}, \tan\theta = \frac{y}{x} = 1$. Since the point is in the third quadrant, the actual angle is, $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

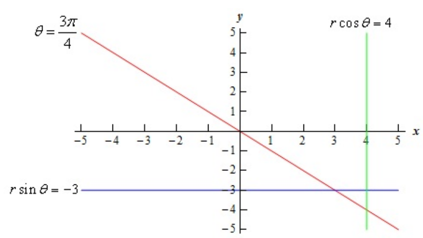
Polar curves

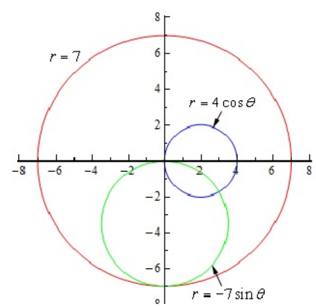
The graph of polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points $P(r, \theta)$ whose coordinates satisfy the equation. Some special cases:
"

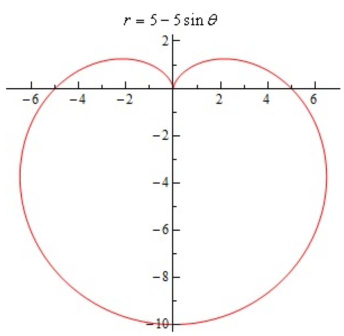
- $\theta = \alpha$: This is a line that goes through the origin and makes an angle of α with the positive x-axis.
- $r\cos\theta = a$: This is equivalent to $x = a$.
- $r\sin\theta = b$: This is equivalent to $y = b$.
- $r = a$: A circle of radius a centered at the origin.

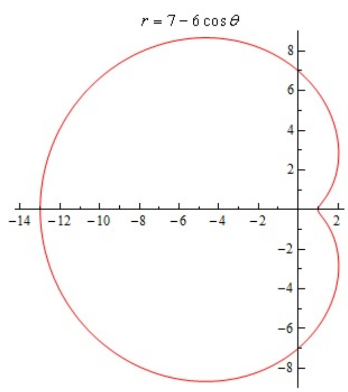
- $r = 2a\cos\theta$: A circle of radius $|a|$ and center $(a, 0)$.
- $r = 2b\sin\theta$: A circle of radius $|b|$ and center $(0, b)$.
- $r = 2a\cos\theta + 2b\sin\theta$: A circle of radius $r = \sqrt{x^2+y^2}$ and center (a, b).

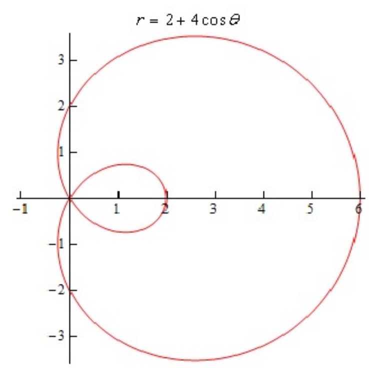
Example 21.











Remark. In the third graph we have an inner loop. To get this, we need to know the value of θ for which the graph will pass through the origin:

$$2 + 4\cos\theta = 0, \Rightarrow \cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

Proposition 1. *Suppose a polar curve is defined by $f(r, \theta) = 0$.*

1. *If $f(r, -\theta) = f(r, \theta)$, the graph is symmetric about the polar axis.*
2. *If $f(-r, \theta) = f(r, \theta)$, the graph is symmetric about the origin.*
3. *If $f(r, \pi - \theta) = f(r, \theta)$, the graph is symmetric about the line $\theta = \frac{\pi}{2}$.*

Tangents to Polar Curves

We want to find the tangent lines to the equation $r = f(\theta)$. From

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta$$

we have

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta}.$$

Example 22. *Find the equation of the tangent line to*

$$r = 3 + 8\sin\theta \ \ at \ \ \theta = \pi/6.$$

Solution:

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta} = \frac{11\sqrt{3}}{5}.$$

Note that at $\theta = \pi/6, r = 7$, which gives $(x, y) = (\frac{7\sqrt{3}}{2}, \frac{7}{2})$. The tangent line is:

$$y = \frac{11\sqrt{3}}{5}x - \frac{98}{5}.$$

Example 23. *Cardioid:* $r = 1 + \sin\theta$. *Find* θ *where we have horizontal or vertical tangent line, or no tangent line.*

Solution:

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{\cos\theta(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}.$$

(1) If the denominator is 0 but the numerator is not 0, we have vertical tangent line. When $sin\theta = 1$, or$sin\theta = 1/2$, the denominator is 0. We have a vertical tangent line when $\theta = \pi/6, 5\pi/6$.

(2) If the denominator is not 0 but the numerator is 0, we have a horizontal tangent line. Hence, we have a horizontal tangent line when $= \pi/2, 7\pi/6, 11\pi/6$.

(3) If the denominator is 0 and the numerator is 0, tangent line does not exist. When $\theta = 3\pi/2$, the tangent line does not exist.

10.4. Areas and Lengths in Polar Coordinates

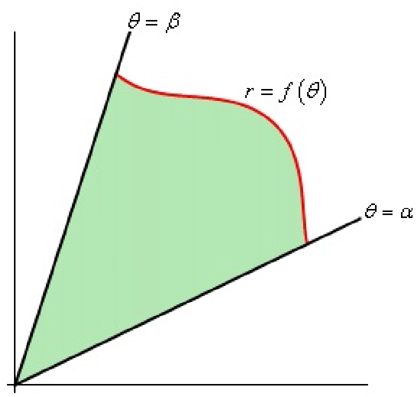
Area

As we know, the area of a sector with radius r and angle θ is

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta.$$

This implies that the area of the following polar region bounded by $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \leq \beta)$ is:

$$A = \int_\alpha^\beta \frac{1}{2}r^2 d\theta.$$



Example 24. *Find the area of the inner loop of $r = 2 + 4\cos\theta$.*

Solution: Let $r = 0$ we have:

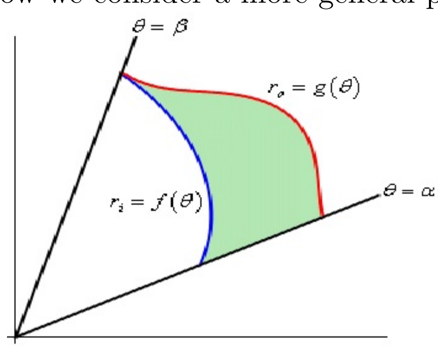
$$2 + 4\cos\theta = 0, \Rightarrow cos\theta = -0.5, \Rightarrow \theta = 2\pi/3, 4\pi/3.$$

So the inner loop is bounded by $r = 2 + 4\cos\theta$ and between $\theta = 2\pi/3$ and $4\pi/3$. Thus

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4\cos\theta)^2 d\theta.$$

$$= \int_{2\pi/3}^{4\pi/3} [6 + 8\cos\theta + 4\cos(2\theta)] d\theta = 4\pi - 6\sqrt{3}.$$

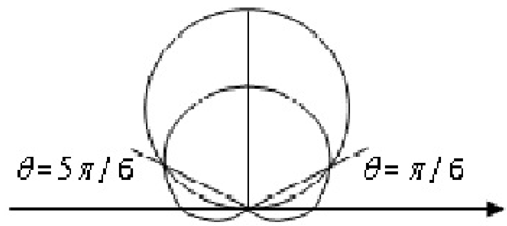
Now we consider a more general polar region:



The area of the shaded part will be :

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_0^2 - r_i^2) d\theta.$$

Example 25. *Find the area of the part outside the cardioid $r = 1 + \sin\theta$, inside the circle $r = 3\sin\theta$.*



Solution: First find intersections: $1+\sin\theta = 3\sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$.

$$A = \int_\alpha^\beta \frac{1}{2}(r_0^2 - r_i^2)d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{2}[(3\sin\theta)^2 - (1+\sin\theta)^2]d\theta = \pi.$$

Example 26. *Find the area of the part outside the cardioid $r = 3+2\sin\theta$, inside the circle $r = 2$.*

Solution: First find intersections: $3+2\sin\theta = 2 \Rightarrow \sin\theta = -1/2 \Rightarrow \theta = 7\pi/6, 11\pi/6$.

$$A = \int_\alpha^\beta \frac{1}{2}(r_0^2 - r_i^2)d\theta = \int_{7\pi/6}^{11\pi/6} \frac{1}{2}[(2)^2 - (3+2\sin\theta)^2]d\theta = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}.$$

Arc length

Now we are going to find the formula for the arc length of the arc $r = f(\theta)$, between $\theta = \alpha$ and $\theta = \beta$ $(\alpha \le \beta)$. Note that

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta, \Rightarrow$$

$$(x'_\theta)^2 + (y'_\theta)^2 = (r'_\theta)^2 + r^2.$$

Thus

$$L = \int_\alpha^\beta \sqrt{(x'_\theta)^2 + (y'_\theta)^2}d\theta = \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2}d\theta.$$

Example 27. *Find the length of the spiral $r = \theta$, $0 \le \theta \le 1$.*

Solution:

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2}d\theta = \int_0^1 \sqrt{\theta^2+1}d\theta \\ &= \int_0^{\pi/4} \sec^3 x dx, \quad \theta = \tan x, d\theta = \sec^2 x dx \\ &= (\sec x \tan x + \ln|\sec x + \tan x|)|_0^{\pi/4} = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2}). \end{aligned}$$

Example 28. *Find the length of the cardioid* $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$.

Solution:

$$
\begin{aligned}
L &= \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2}\,d\theta = \int_0^{2\pi} \sqrt{2(\sin\theta + 1)}\,d\theta \\
&= \sqrt{2}\int_0^{2\pi} |\sin\frac{\theta}{2} + \cos\frac{\theta}{2}|\,d\theta = 2\sqrt{2}\int_0^{\pi} |\sin\varphi + \cos\varphi|\,d\varphi, \quad \varphi = \theta/2 \\
&= 2\sqrt{2}\int_0^{3\pi/4} (\sin\varphi + \cos\varphi)\,d\varphi - 2\sqrt{2}\int_{3\pi/4}^{\pi} (\sin\varphi + \cos\varphi)\,d\varphi \\
&= 8.
\end{aligned}
$$

Chapter 12: Vectors and the Geometry of Space

12.1. Three-Dimensional Coordinate Systems

- The three-dimensional Cartesian (Rectangular) coordinate system (x, y, z).
- Planes parallel to the coordinate planes: $x = a$ is a plane parallel to yz-plane, similarly we have $y = b, z = c$.
- Distance formula in 3-D: Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$. Then

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- Equation of a sphere with centre (x_0, y_0, z_0) and radius r is:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

12.2. Vectors

Algebraic representation of vectors:

- Vectors in $\mathbb{R}^2$: $\vec{v} = \mathbf{v} = (a, b) = \begin{bmatrix} a \\ b \end{bmatrix}$, zero vector $\vec{0} = (0, 0)$.
- Vectors in $\mathbb{R}^3$: $\vec{v} = (a, b, c) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, zero vector $\vec{0} = (0, 0, 0)$.
- Length (norm, magnitude) $||(a, b)|| = \sqrt{a^2 + b^2}$, $||(a, b, c)|| = \sqrt{a^2 + b^2 + c^2}$.
- Sum: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.
- Scalar multiple: Let $\vec{u} = (u_1, u_2, u_3)$, c be a scalar, then $c\vec{u} = (cu_1, cu_2, cu_3)$.
- Distance: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$.

- unit vector: $||\vec{u}|| = 1$.
- Standard basis vectors: $\vec{i} =< 1,0,0 >, \vec{j} =< 0,1,0 >, \vec{k} =< 0,0,1 >$. Position vectors can be expressed in terms of standard basis vectors: $< a,b,c >= a\vec{i} + b\vec{j} + c\vec{k}$.

Example 29. *Let $\vec{u} = (1,2,-2)$. Find the unit vector which has the same direction as $\vec{u}$.*

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$, $\vec{u} + \vec{0} = \vec{u}$, $\vec{u} + (-\vec{u}) = \vec{0}$;
- $(cd)\vec{u} = c(d\vec{u})$, $(c+d)\vec{u} = c\vec{u} + d\vec{u}$, $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$;
- $1\vec{u} = \vec{u}$, $(-1)\vec{u} = -\vec{u}$, $0\vec{u} = \vec{0}$, $\vec{u} // \vec{v} \Leftrightarrow \vec{v} = c\vec{u}$.

12.3. The Dot Product

- Dot product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \le \theta \le \pi$, then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$.
- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.
- Direction angles to the three axis and direction cosines of vectors.:

$$\cos\alpha = \frac{\vec{u}\cdot\vec{i}}{||\vec{u}||\,||\vec{i}||}, \quad \cos\beta = \frac{\vec{u}\cdot\vec{j}}{||\vec{u}||\,||\vec{j}||}, \quad \cos\gamma = \frac{\vec{u}\cdot\vec{k}}{||\vec{u}||\,||\vec{k}||}.$$

  They satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \frac{\vec{u}}{||\vec{u}||} = (\cos\alpha, \cos\beta, \cos\gamma).$$

- Projection: The projection of $\vec{u}$ onto $\vec{v}$ is

$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\right)\vec{v}, \quad comp_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}.$$

Example 30. *Let $\vec{u} = (1,2,-2)$, $\vec{v} = (-2,-2,1)$, then $\vec{u} \cdot \vec{v} = -8$.*

Example 31. *Let $\vec{u} = (1,2,-2)$, $\vec{v} = (-2,-2,1)$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Solution:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||} = \frac{-8}{9}.$$

Properties: Let c be a scalar.

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $\vec{w}\cdot(\vec{u}+\vec{v}) = \vec{w}\cdot\vec{u} + \vec{w}\cdot\vec{v}$
- $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
- $\vec{u}\cdot\vec{0} = 0$
- $\vec{u}\cdot\vec{u} = ||\vec{u}||^2$.

12.4. The Cross Product

- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2, +u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

- Orthogonal: $\vec{u}\times\vec{v}\perp\vec{u}$, $\vec{u}\times\vec{v}\perp\vec{v}$.

Example 32. *Find a vector that is orthogonal to both* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: Any scalar multiple of $\vec{u}\times\vec{v} = (8, -3, 2)$.
Properties: Let c be a scalar.

- $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$
- $\vec{w}\times(\vec{u}+\vec{v}) = \vec{w}\times\vec{u} + \vec{w}\times\vec{v}$
- $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
- $c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times(c\vec{v})$
- $\vec{u}\times\vec{0} = \vec{0}$
- $\vec{u}\times\vec{u} = \vec{0}$
- $||\vec{u}\times\vec{v}|| = ||\vec{u}||\,||\vec{u}||\,\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$
- $||\vec{u}\times\vec{v}||$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example 33. *Find the area of the parallelogram determined by* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: $A = ||\vec{u} \times \vec{v}|| = ||(8, -3, 2)|| = \sqrt{77}$.

Example 34. *Find the area of the triangle with vertices* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$A = \frac{1}{2}||\vec{PQ} \times \vec{PR}|| = ||(4, 6, -8)|| = \sqrt{29}.$$

12.5 Equations of Lines and Planes

Line:

A line is determined by a point and a vector (direction vector) parallel to the line. Let $P(p_1, p_2, p_3)$ be a point on the line L. Let $\vec{v}$ be a nonzero vector which is parallel L.

- Point-parallel form (vector form): $\vec{x}(t) = \vec{p} + t\vec{v}$, $t \in \mathbb{R}$, $\vec{p} = (p_1, p_2, p_3)$, $\vec{x}(t) = (x, y, z)$.
- Parametric form: $x = p_1 + tv_1$, $y = p_2 + tv_2$, $z = p_3 + tv_3$.
- Symmetric form: $\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$.

Remark. (Two-point form). If a line goes through two points P and Q, then $\vec{x}(t) = \vec{p} + t(\vec{q} - \vec{p})$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P.

Example 35. *Find the equation of the line through* $P(1, 2, 3)$ *and* $Q(3, 1, 1)$.

Two lines L_1 and L_2 can be

- parallel
- intersected
- skewed

Example 36. *Show that the intersection between* $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t$ *and* $L_2 : x = 2s, y = 3 + s, z = -3 + 3s$ *is* $(16/5, 23/5, 9/5)$.

Example 37. *Show that the two lines* $L_1 : x = 1 + t, y = -2 + 3t, z = 4 - t; L_2 : x = 2s, y = 3 + s, z = -3 + 4s$ *are skew lines.*

Solution. (1) The two direction vectors are $v_1 =< 1, 3, -1 >$ and $v_2 =< 2, 1, 4 >$. They are not parallel.

(2) No intersection: From "$x = x$" and "$y = y$" we have $1 + t = 2s, -2 + 3t = 3 + s \Rightarrow s = 1.6, t = 2.2$. By L_1, $z = 1.8$; by L_2, $z = 3.4$.

Plane:

A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point on the plane. Let $\vec{n}$ be a nonzero vector which is perpendicular to the plane.

- Point-normal form: $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$, $\vec{x} = (x, y, z)$.
- Parametric form: $n_1(x - p_1) + n_2(y - p_2) + n_3(zp_3) = 0$, where $(n_1, n_2, n_3) = \vec{n}$.
- Standard form (linear equation): $n_1x + n_2y + n_3z = d$, where $d = n_1p_1 + n_2p_2 + n_3p_3$.

Example 38. *Find the equation of the plane through three points $P(1, 2, 3)$, $Q(-3, 2, 1)$, and $R(2, 4, 5)$.*

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$. $\vec{n} = \vec{PQ} \times \vec{PR} = (4, 6, -8)$. Thus

$$4(x - 1) + 6(y - 2) - 8(z - 3) = 0, \Rightarrow 4x + 6y - 8z = -8.$$

Example 39. *Find the intersection between the line $L : x = 1 + t, y = -2 + 3t, z = 4 - t$ and the plane $3x + 5y + 8z = 1$.*

Angle between two planes:

Two planes are parallel if their normal vectors are parallel. The angle between two planes is defined as the angle between their normal vectors:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\cdot \vec{n_2}|}.$$

Example 40. *Find the angle between two planes $x - 3y - 2z = 3$ and $3x + 5y + 8z = 1$.*

Solution: $\cos\theta = \frac{-2}{\sqrt{7}}, \Rightarrow \theta = 180^o - 41^o = 139^o$.

Intersection between two planes:

If the two planes with normal vectors $\vec{n_1}$ and $\vec{n_2}$ are not parallel, then the intersection is a line with direction vector $\vec{n_1} \times \vec{n_2}$.

Example 41. *Find the angle between two planes $x - 3y - 2z = 2$ and $2x + y + 3z = 1$.*

Solution: (1) $\vec{n_1} \times \vec{n_2} =< -7, 7, 7 >= 7 < -1, 1, 1 >$.

(2) To find one intersection point, we let $z = 0$. Then $x = 5/7, y = -3/7$. So the parametric equation of the line is:

$$x = 5/7 - t, y = -3/7 + t, z = t.$$

Distance between a point and a plane:

A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane. Let $P(p_1, p_2, p_3)$ be a point, let Π be: $ax + by + cz = d$. Then the distance between them is:

$$D = \frac{|ap_1 + bp_2 + cp_3 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 42. *Show that the distance between $P(1, 2, 3)$ and the plane $2x - 2y - z = 1$ is 2.*

Remark. Finding the distance between two parallel planes: This equals the distance from any point in one plane to the other plane; Finding the distance from a point P to a line L; Finding distance between two skew lines.

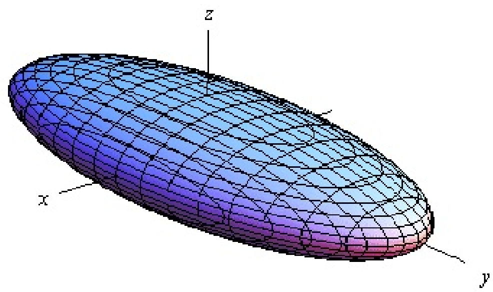
12.6 Cylinders and Quadric Surfaces

A **quadric surface** is the graph of a second degree equation in three variables x, y, z:

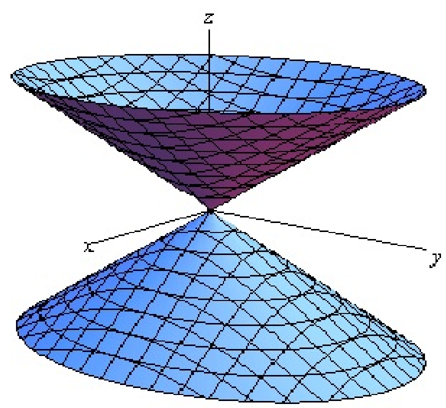
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

By rotating the surface, or equivalently, rotating the axes, we may assume that $D = E = F = 0$.

1. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



2. Cone: The general equation of a cone is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, $a, b, c > 0$.



3. Cylinder

A cylinder is a surface consisting of lines parallel (rulings) to a given line passing through a given plane curve. In most cases, the rulings are parallel to an axis. In this case, the equation of the surface contains only two variables, which gives the plane curve in a coordinate plane. For example,

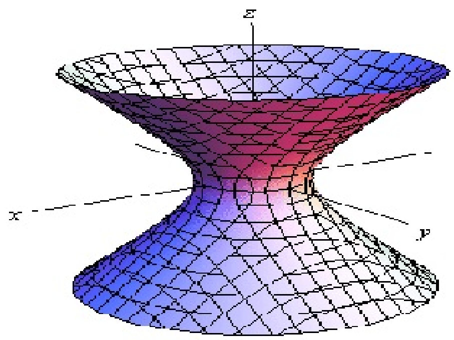
(i) $x^2 + y^2 = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the unit circle in the xy- plane.

(ii) $z = x^2$. This is a cylinder containing lines parallel to the y-axis passing through points on the curve $z = x^2$ in the xz- plane.

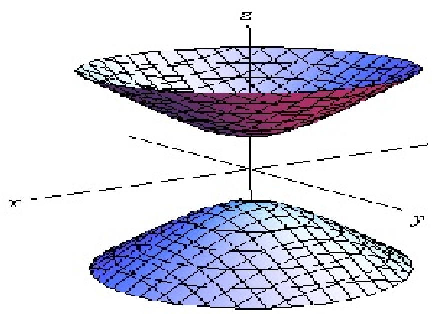
(iii) $yz = 1$. This is a cylinder containing lines parallel to the x-axis passing through points on two branches of the curve $yz = 1$ in the yz- plane.

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is a cylinder containing lines parallel to the z-axis passing through points on the ellipse in the xy- plane.

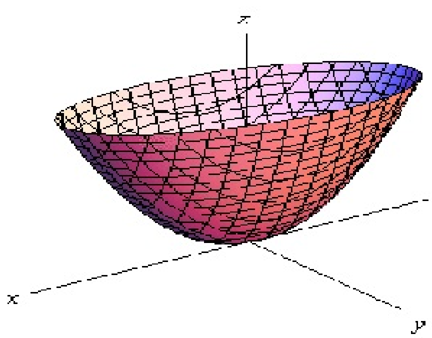
4. Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



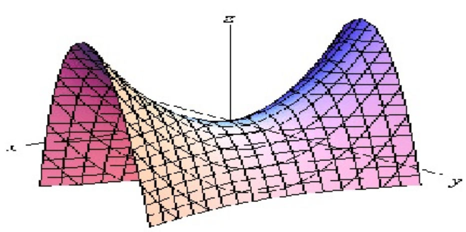
5. Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.



6. Elliptic Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



7. Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$, $a, b, c > 0$.



Example 43. *Classify the following quadric surfaces:*

(1) $z = -x^2 - y^2 + 4$. This is an elliptic paraboloid that opens downward.

(2) $4x^2 - y^2 + z^2 - 8x - 2y - 6z - 4 = 0$. This is Hyperboloid of One Sheet with centre (1,-1,3), and centered on the line which is parallel to z-axis.

(3) $4x^2 - y^2 + 2z^2 + 4 = 0$. This is a hyperboloid of two sheets centered at the x-axis.

(4) $x^2 + 2z^2 - 6x - y + 10 = 0$. This is an elliptic paraboloid with vertex (3, 1, 0), centered with line x = 3, y = 1.

Remark. If a quadratic equation of x, y, and z with terms xy, yz, or xz, then a rotation is needed to convert it to the standard form.

Chapter 13: Vector Functions

13.1 Vector Functions and Space Curves

A vector function (or vector-valued function) is a function that takes one or more variables and returns a vector. Here we consider 3-dimensional vectors:

$$\vec{r}(t) = < f(t), g(t), h(t) > = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k},$$

where $f(t), g(t), h(t)$ are called component functions. The domain is $D(r) = D(f) \bigcap D(g) \bigcap D(h)$. If $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$, then we say that $\vec{r}(t)$ is continuous at $t = a$, where

$$\lim_{t\to a} \vec{r}(t) = < \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) > .$$

Example 44. *Let* $\vec{r}(t) = (\sin t/t)\vec{i} + \sqrt{t}\vec{j} + \ln(5-t)\vec{k}$, *then* $D(\vec{r}) = (0, 5)$.

$$\lim_{t\to 0} \vec{r}(t) = \vec{i} + \ln 5\vec{k}.$$

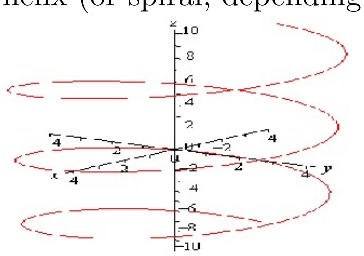
Definition 1. *The set C of all points (x, y, z) is called a space curve, where $x = f(t), y = g(t), z = h(t)$ are called parametric equations of C and t varies throughout an interval I.*

Example 45. *Sketch the graph of the following vector function:* $\vec{r}(t) = < 3-2t, 2, 3-5t >$.

Solution:$\vec{r}(t) = < 3, 2, 3 > +t < -2, 0, -5 >$, a line.

Example 46. *Sketch the graph of the following vector function:* $\vec{r}(t) = < 4\cos t, 4\sin t, t >$.

Solution. The x and y coordinates will form a circle centered on the z-axis. As we increase t the circle that is being traced out in the x and y directions should be rising. So, we've got a helix (or spiral, depending on what you want to call it) here.



Example 47. *Find a vector equation that represents the intersection between a cylinder* $\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$ *and the plane* $y - z = 3$.

Solution: The intersection is a curve C, whose projection onto xy-plane is an ellipse. Let $x = 2\cos t, y = 4sint$, $0 \le t \le 2\pi$. Then $z = y - 3 = 4\sin t - 3$. Thus

$$\vec{r}(t) = 2\cos t\vec{i} + 4\sin t\vec{j} + (4sint - 3)\vec{k}, 0 \le t \le 2\pi,$$

which is called parametrization of C.

13.2 Derivatives and integrals of vector equations

Derivatives:

$$\vec{r}'(t) =< f'(t), g'(t), h'(t) >= f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k},$$

which is also called tangent vector to the curve defined by $\vec{r}(t)$. If $\vec{r}'(t)$ exists and $\vec{r}'(t) \neq 0$, the tangent line to C through a point P is the line through P and parallel to $\vec{r}'(t)$. Unit tangent vector is $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$. As usual, $\vec{r}''(t) = [\vec{r}'(t)]'$.

Differentiation rules:

(1) $[\vec{u}(t) \pm \vec{v}(t)]' = \vec{u}'(t) \pm \vec{v}'(t)$.

(2) $[c\vec{u}(t)]' = c\vec{u}'(t)$.

(3) $[f(t)\vec{u}(t)]' = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$.

(4) $[\vec{u}(t) \cdot \vec{v}(t)]' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$. (Note that $\vec{u}(t) \cdot \vec{v}(t)$ is a scalar function of t)

(5) $[\vec{u}(t) \times \vec{v}(t)]' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$.

(6) $[\vec{u}(f(t))]' = \vec{u}'(f(t))(f'(t))$. (Chain rule)

Example 48. *Let $\vec{r}(t) = t^6\vec{i} + \sqrt{t}\sin(2t)\vec{j} + \ln(1+t)\vec{k}$. Find the parametric equations of the tangent line to the curve at the point* $(0, 0, 0)$.

Solution:

$$\vec{r}'(t) = 6t^5\vec{i} + 2\cos(2t)\vec{j} - \frac{1}{t+1}\vec{k}, \Rightarrow \vec{r}'(0) = 2\vec{j} - \vec{k}.$$

The tangent line is:

$$\vec{r}(t) =< 0, 0, 0 > +t < 0, 2, -1 >, \text{ or } x = 0, y = 2t, z = -t.$$

Integrals:

Antiderivative:

$$\int \vec{r}(t)dt =< \int f(t)dt, \int g(t)dt, \int h(t)dt >= \left(\int f(t)dt\right)\vec{i} + \left(\int g(t)dt\right)\vec{j} + \left(\int h(t)dt\right)\vec{k}.$$

Definite Integral:

$$\int_a^b \vec{r}(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt >= \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}.$$

Example 49. *Let $\vec{r}(t) = (2\cos t, \sin t, 2t)$. Then*

$$\int \vec{r}(t)dt =< \int f(t)dt, \int g(t)dt, \int h(t)dt >= \left(\int f(t)dt\right)\vec{i} + \left(\int g(t)dt\right)\vec{j} + \left(\int h(t)dt\right)\vec{k}$$

$$= (2\sin t)\vec{i} - (\cos t)\vec{j} + (t^2)\vec{k} + \vec{c},$$

$$\int_0^{\pi/2} \vec{r}(t)dt = 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}.$$

13.3 Arc Length and Curvature

Arc Length: The length of the arc $\vec{r}(t) = < f(t), g(t), h(t) > = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}, a \leq t \leq b$ is :

$$L = \int_a^b |\vec{r}'(t)|dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt.$$

Example 50. *Find the length of the curve* $\vec{r}(t) = 8t\vec{i} + 3\sin(2t)\vec{j} + 3\cos(2t)\vec{k}, 0 \leq t \leq 2\pi$.

Solution:

$$\vec{r}'(t) = 8\vec{i} + 6\cos(2t)\vec{j} - 6\sin(2t)\vec{k}, \Rightarrow |\vec{r}'(t)| = 10, \Rightarrow L = \int_0^{2\pi} |\vec{r}'(t)|dt = \int_0^{2\pi} 10dt = 20\pi.$$

We call

$$s = s(t) = \int_a^t |\vec{r}'(u)|du$$

the arc length function. We have

$$s'(t) = |\vec{r}'(t)|.$$

Example 51. *Given the curve* $\vec{r}(t) = 8t\vec{i} + 3\sin(2t)\vec{j} + 3\cos(2t)\vec{k}$.

(1) Find the arc length function. Solution:

$$s = s(t) = \int_0^t |\vec{r}'(u)|du = 10t.$$

(2) Reparametrize the curve $\vec{r}(t)$ with respect to arc length measured from (0,0,3).
Solution: $s = 10t, sot = 0.1s$,

$$\vec{r}(s) = \vec{r}(t(s)) = 0.8s\vec{i} + 3\sin(0.2s)\vec{j} + 3\cos(0.2s)\vec{k}.$$

(3) Where on the curve are we after traveling for a distance of 5π?
Solution: $s = 5\pi$,

$$\vec{r}(s) = \vec{r}(t(s)) = 0.8s\vec{i} + 3\sin(0.2s)\vec{j} + 3\cos(0.2s)\vec{k} = 4\pi\vec{i} + 3\sin(\pi)\vec{j} + 3\cos(\pi)\vec{k}.$$

So we are at $(4\pi, 0, -3)$.

Curvature:

The curvature measures how fast a curve is changing direction at a given point. The curvature of a curve $\vec{r}(t)$ is:

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}, \quad \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

Example 52. *The curvature of a circle of radius r is 1 / r.*

Example 53. *Let* $\vec{r}(t) =< t, t^2, t^3 >$. *Find* $\kappa(t)$.

Solution: $\vec{r}'(t) =< 1, 2t, 3t^2 >, \vec{r}''(t) =< 0, 2, 6t > \Rightarrow$

$$|\vec{r}' \times \vec{r}''| = | < 6t^2, -6t, 2 > | = 2\sqrt{9t^4 + 9t^2 + 1}, \Rightarrow$$

$$\kappa(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\sqrt{9t^4 + 4t^2 + 1}^3}.$$

Example 54. *Let* $\vec{r}(t) =< \cos t, \sin t, t >$. *Then* $\kappa(t) = \frac{1}{2}$.

Special case, if the curve is in the xy- plane, with $y = f(x)$, then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

Example 55. *Find x where the graph of* $y = e^x$ *has the maximum curvature.*

Solution:

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{e^x}{[1 + e^{2x}]^{3/2}}, \Rightarrow$$

$$\kappa'(x) = \frac{e^x(1 + e^{2x})^{1/2}(1 - 2e^{2x})}{[1 + e^{2x}]^3} = 0 \Rightarrow x = -\frac{\ln 2}{2}.$$

The Normal and Binormal Vectors:

Consider a smooth curve C defined by a vector function $\vec{r}(t)$, i.e., $\vec{r}'(t) \neq 0$. Since $\vec{T}$ and $\vec{T}'$ are orthogonal, $\vec{T}'$ gives the direction that is normal (i.e., perpendicular) to this curve.

- Principal unit normal vector, or unit normal: $\vec{N} = \vec{T}'/|\vec{T}'|$, which is a unit vector in the direction of $\vec{T}'$.

- Binormal vector: $\vec{B} = \vec{T} \times \vec{N}$ (which is also a unit vector).

- Osculating circle: Let $P = \vec{r}(t_0)$ be a point on the curve C. Let κ be the curvature at P, where $\kappa \neq 0$. Let Q be a point at distance $\rho = \frac{1}{\kappa}$ along $\vec{N}$, in the same direction if κ is positive and in the opposite direction if κ is negative. The circle with center at Q and with radius ρ is called the **osculating circle** to the curve C at the point P. We can use the osculating circle to approximate a small segment of a curve defined by a vector function $\vec{r}(t)$ around $\vec{r}(t_0)$.

- Normal plane: The plane determined by the normal vector $\vec{N}$ and the binormal vector $\vec{B}$ at a point P on a curve C is called normal plane, which is perpendicular to $\vec{T}$. The equation of the normal plane at point $\vec{r}(t_0)$ is

$$\vec{T} \cdot (\vec{r}(t) - \vec{r}(t_0)) = 0.$$

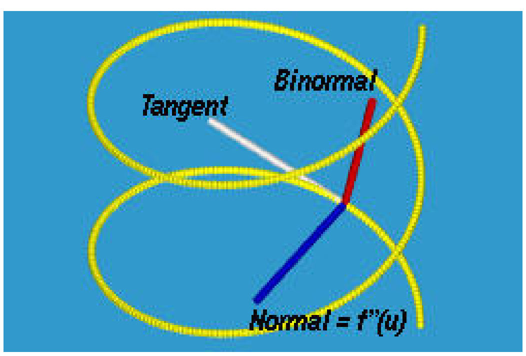
- Osculating plane: The plane determined by the normal vector $\vec{N}$ and the tangent vector $\vec{T}$ at a point P on a curve C is called osculating plane, which is perpendicular to $\vec{B}$. The equation of the normal plane at point $\vec{r}(t_0)$ is

$$\vec{B} \cdot (\vec{r}(t) - \vec{r}(t_0)) = 0.$$

  The osculating circle is in an osculating plane.

- $\vec{N}$ is perpendicular to the tangent line of the curve, pointing to the "concave side" of the curve, i.e., the side where the center of the osculating circle is. The center of the osculating circle is on vector $\vec{N}$. Hence, the center of the osculating circle is

$$\vec{r}(t_0) + \frac{1}{\kappa(t_0)}\vec{N}.$$



Example 56. *Let $\vec{r} = (\cos t, \sin t, t)$. Find unit normal, binormal, the equation of the osculating plane, the equation of the osculating circle, and the equation of the normal plane at t_0.*

Solution:

$$\vec{r}\,'(t) = (-\sin t, \cos t, 1), \Rightarrow |\vec{r}\,'(t)| = \sqrt{2}.$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1), \Rightarrow \vec{T}\,'(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0), \Rightarrow |\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}.$$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} = (-\cos t, -\sin t, 0),$$

$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1).$$

The equation of the osculating plane at t_0 is

$$\vec{B} \cdot (\vec{r}(t) - \vec{r}(t_0)) = 0 \Rightarrow (\sin t_0)x - (\cos t_0)y + z = t_0.$$

The radius of the osculating circle is $\frac{1}{\kappa(t_0)} = 2$. The center of the osculating circle is at

$$\vec{r}(t_0) + \frac{1}{\kappa(t_0)}\vec{N} = (-\cos t_0, -\sin t_0, t_0).$$

The vector form of the normal plane is

$$\vec{T} \cdot (\vec{r}(t) - \vec{r}(t_0)) = 0 \Rightarrow -(\sin t_0)x + (\cos t_0)y + z = t_0.$$

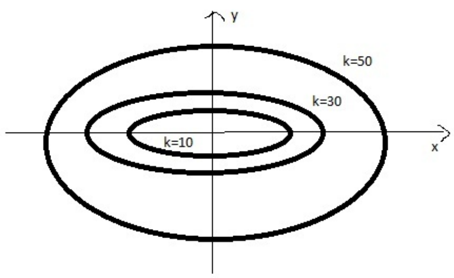
Chapter 14: Partial Derivatives

14.1. Functions of Several Variables

Functions of two variables: $z = f(x, y)$.
Level curves (contour maps) of $f(x, y)$: $f(x, y) = k$ for different k.
Example. Sketch three level curves:

1. $f(x, y) = 2x + 3y$;

2. $f(x, y) = 2x^2 + 20y^2$.



3. $f(x, y) = x^2 - y^2$.

4. $f(x, y) = x^2 - y$.

Functions of three variables: A function of three variables $w = f(x, y, z)$ is a rule which maps each point (x,y,z) in a set D to a unique number w. The set D is called the domain of the function, which is often denoted D(f).
Example. Find the domain of the function $f(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$.

Solution: $4 - x^2 - y^2 - z^2 \geq 0$, i.e., $x^2 + y^2 + z^2 \leq 4$. Thus $D = \{(x, y) : x^2 + y^2 + z^2 \leq 4\}$.
$w = f(x, y, z)$.
Level surfaces of $f(x, y, z)$: $f(x, y, z) = k$ for different k.

14.2. Limits and Continuity

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

In general, if there are different limits when (x, y) approaches (a, b) along different paths, then the limit does not exist. $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

14.3. Partial Derivatives

Functions of two variables

- Partial derivatives of $z = f(x, y)$:

  $$z_x = \frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

  which is the derivative of f with respect to x;

  $$z_y = \frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h},$$

  which is the derivative of f with respect to y.

- Methods:

  1. To find f_x: regard y as a constant, and differentiate $f(x, y)$ with respect to x;
  2. To find f_y: regard x as a constant, and differentiate $f(x, y)$ with respect to y.

- Meaning: f_x means the rate of change of f with respect to x when y is fixed.

Example. Let $f(x, y) = e^{xy} + \frac{x}{y}$. Calculate $f_x(0, 1)$, $f_y(0, 1)$.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(0, 1) = 2.$$

$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(0, 1) = 0.$$

Functions of three variables

- Let $w = f(x, y, z)$, then

  $$\frac{\partial w}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y, z) := D_x f := \lim_{h\to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h},$$

  which is the derivative of f with respect to x.

- Meaning: f_x means the rate of change of f (or w) with respect to x when y and z are fixed.

Example. Let $f(x,y,z) = \sin z e^{xy} \ln x$. Calculate f_x, f_y, f_z.

Implicit differentiation:

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if z is implicitly defined by

$$x^2 + y^3 + z^4 - 8xyz = 1.$$

Higher derivatives:

f_{xx}, $\frac{\partial^3 f}{\partial z \partial y \partial x} = f_{xyz}$,...

Example. Let $f(x,y) = e^{xy} + \frac{x}{y}$. Calculate f_{xx}, f_{xy}, f_{yy}.

Solution:

$$f_x = ye^{xy} + \frac{1}{y}, f_y = xe^{xy} - \frac{x}{y^2}.$$

$$f_{xx} = y^2 e^{xy}, \quad f_{xy} = e^{xy} + xye^{xy} - \frac{1}{y^2}, \quad f_{yy} = x^2 e^{xy} + \frac{2x}{y^3}.$$

Example. Let $f(x,y,z) = \sin z e^{xy} \ln x$. Calculate f_{xyz}.

14.4. Tangent Plane and Linear Approximation

1. Functions of two variables.

- Equation of the tangent plane of $z = f(x,y)$ at (x_0, y_0):

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Linear Approximation (Tangent plane approximation):

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= L(x,y). \end{aligned}$$

- Differential notation: $dz = f_x(x,y)dx + f_y(x,y)dy$. If we take $dx = x - x_0$, $dy = y - y_0$, then $dz|_{(x_0,y_0)} = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

Example. Find the equation of the tangent plane of the surface $z = e^{xy} + \frac{x}{y}$ at the point $(2, 1, e^2 + 2)$.

Solution. Let $f(x,y) = e^{xy} + \frac{x}{y}$. Then

$$f_x = ye^{xy} + \frac{1}{y}, \quad f_x(2,1) = e^2 + 1.$$

$$f_y = xe^{xy} - \frac{x}{y^2}, \quad f_y(2,1) = 2e^2 - 2.$$

Thus the equation of the tangent plane at the point $(2, 1, e^2 + 2)$ is

$$z - (e^2+2) = (e^2+1)(x-2) + (2e^2-2)(y-1), \quad i.e., \quad z = (e^2+1)x + (2e^2-2)y - 3e^2 + 2.$$

Example. Use the linear approximation of $f(x,y) = 2x^2y^2 + 3xy + x$ at $(1,1)$ to approximate $f(0.9, 1.1)$.

Solution.
$f(x,y) \approx f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$.
$f(x,y) = 2x^2y^2 + 3xy + x,\ f_x = 4xy^2 + 3y + 1,\ f_y = 4x^2y + 3y$
$f(1,1) = 6,\ f_x(1,1) = 8,\ f_y(1,1) = 7$
Thus $f(x,y) \approx 6 + 8(x-1) + 7(y-1)$.
$f(0.9,\ 1.1) \approx 6 + 8*(-0.1) + 7*0.1 = 5.9$

Example. Let $z = x^2 + 3xy - y^2$.

(a) Find dz.

(b) If x changes from 2 to 2.05, y changes from 3 to 2.96, calculate dz and Δz.

Solution: (a) $dz = f_x(x,y)dx + f_y(x,y)dy = (2x+3y)dx + (3x-2y)dy$.

(b) Putting $x_0 = 2$, $dx = 2 - 2.05 - 2 = 0.05$, $y_0 = 3$, $dy = 2.96 - 3 = -0.04$, then

$$dz|_{(x_0,y_0)} = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0.65.$$

$$\Delta z = f(2.05, 2.96) - f(2,3) = 0.6449.$$

2. Functions of three or more variables $w = f(x,y,z)$.

- Linear Approximation (Tangent plane approximation) at (x_0, y_0, z_0):

$$\begin{aligned} f(x,y,z) &\approx f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) \\ &\quad + f_z(x_0,y_0,z_0)(z-z_0) \\ &= L(x,y,z). \end{aligned}$$

- Equation of the tangent plane of to the level surface $f(x,y,z) = k$ at (x_0, y_0, z_0) is: $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$.

- Differential: $dw = w_x dx + w_y dy + w_z dz$.

Example. A rectangular box has measured dimensions $75 \times 60 \times 40$. The maximum error of the measurement to each side is 0.1. Find the maximum error of the volume.

Solution: Let $V = xyz$. Then

$$dV = yzdx + xzdy + xydz, \Rightarrow \Delta V \doteq dV = 990.$$

14.5. The Chain Rule

- Basic Chain Rule

  1. If $z = f(x,y)$, $x = g(t)$, $y = h(t)$, then
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
  2. $\frac{dz}{dt}$ means rate of change of z with respect to t along the path $x = g(t)$, $y = h(t)$, $t \in D$.
  3. If $w = f(x,y,z)$, $x = g(t)$, $y = h(t)$, $z = k(t)$, then
  $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

- General Chain Rule: If $w = f(x,y,z)$, $x = g(u,v)$, $y = h(u,v)$, $z = k(u,v)$, then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

- Implicit Differentiation:

  1. If $F(x,y)=0$, then
  $$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$
  Here when we calculate partial derivatives, we consider x and y as independent variables.

  2. If $F(x,y,z)=0$, then
  $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$
  Here when we calculate partial derivatives, we consider x, y and z as independent variables.

Example. Suppose $z=f(x,y)$ where $x=g(t)$ and $y=h(t)$. Given the data

$g(1)=1,\ g'(1)=2,$

$h(1)=2, h'(1)=3,$

$f_x(1,2)=-1,\ f_y(1,2)=2.$

Find $\frac{dz}{dt}$ when $t=1$.

Solution.
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x g' + f_y h'$$
$$t=1 \Rightarrow (x,y) = (g(1),h(1)) = (1,2)$$
$$\frac{dz}{dt}|_{t=1} = f_x(1,2)g'(1) + f_y(1,2)h'(1)$$
$$= (-1)(2)+(2)(3) = 4$$

Example. Consider the following function
$$z = x^2y + e^x\cos y, \quad x = t^3\sin s,\ y = s^2+3t^2.$$
Calculate $\frac{\partial z}{\partial s}$ at the point $(s,t)=(0,1)$ by using Chain Rule.

Example. Find y' if $x^2y+e^x\cos y = 3$.

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2y^3+z^4+5xyz=3$.

The normal form of the tangent plane to $F(x,y,z)=0$:
$$< F_x, F_y, F_z > \cdot < x-x_0, y-y_0, z-z_0 >= 0.$$

Example 57. *Given $x^3+y^3+z^3+6xyz+4=0$. Find the tangent plane at the point (1, 1, 2).*

Solution : The tangent plane at (1, 1, 2) is $-9(x-1)+15(y+1)+6(z-2)=0$, or $3x+5y+2z+4=0$.

14.6. Directional Derivatives and the Gradient Vector

14.6.1 Gradients and Directional Derivatives in the Plane

- Directional derivatives:

  1. The directional derivative of the function $f(x,y)$ at (x_0,y_0) in the direction of a unit vector $\vec{u}=<u_1,u_2>$ is

$$\begin{aligned} D_{\vec{u}}f(x_0,y_0) &= \lim_{h\to 0}\frac{f(x_0+hu_1,y_0+hu_2)-f(x_0,y_0)}{h} \\ &= f_x(x_0,y_0)u_1+f_y(x_0,y_0)u_2 \\ &= <f_x(x_0,y_0),f_y(x_0,y_0)>\cdot<u_1,u_2>. \end{aligned}$$

  2. $D_{\vec{u}}f(x_0,y_0)$ means the rate of change of $f(x,y)$ at (x_0,y_0) in the direction of $\vec{u}$.

- The gradient of $f(x,y)$ at (x_0,y_0) is

$$\nabla f(x_0,y_0)=gradf(x_0,y_0)=<f_x(x_0,y_0),f_y(x_0,y_0)>.$$

  1. $\nabla f(x_0,y_0)$ points into the direction of maximum increase of f at (x_0,y_0).
  2. $\nabla f(x_0,y_0)$ is perpendicular to the contour line (or level curve) of f through (x_0,y_0).
  3. $|\nabla f(x_0,y_0)|$ is the maximum rate of change of f at (x_0,y_0).

Example. Let $f(x,y)=\sqrt{2x^2+3y^2}$.

(1) Sketch three level curves, and calculate the gradient of the function.

(2) Find the directional derivative of $f(x,y)$ at the point $(2,1)$ in the direction of the vector $\langle 1,\sqrt{3}\rangle$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Example. Let $f(x,y)=x^2y+4y^2$.

(1) Calculate the gradient of the function.

(2) Find the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction of the vector $\langle 1, \sqrt{3}\,\rangle$.

(3) Find the maximum rate of change of f at (2, 1) and indicate in which direction this maximum will occur.

Solution. (1) $\nabla f = \langle 2xy, x^2 + 8y\rangle$.

(2) $\vec{v} = \langle 1, \sqrt{3}\,\rangle$, $\vec{u} = \frac{1}{2}\langle 1, \sqrt{3}\,\rangle$,, $\nabla f(2, 1) = \langle 4, 12\rangle$.

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$D_{\text{u}} f(2, 1) = \nabla f(2, 1) \cdot \vec{u} = \langle 4, 12\rangle \cdot \frac{1}{2}\langle 1, \sqrt{3}\,\rangle = 2 + 6\sqrt{3}$

(3) The maximum rate of change of f at (2, 1) $= |\nabla f(2, 1)| = |\langle 4, 12\rangle| = 4\sqrt{10}$, which occurs in the direction $\langle 4, 12\rangle$.

14.6.2 Gradient and Directional Derivatives in space

- Directional derivatives:

  1. The directional derivative of the function $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} =< u_1, u_2, u_3 >$ is

  $$f_{\vec{u}}(x_0, y_0, z_0) = D_{\vec{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3.$$

- The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is

  $$grad f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) =< f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) > .$$

  1. $\nabla f(x_0, y_0, z_0)$ points into the direction of maximum increase of f at (x_0, y_0, z_0).
  2. $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface of f through (x_0, y_0, z_0).
  3. $|\nabla f(x_0, y_0, z_0)|$ is the maximum rate of change of f at (x_0, y_0, z_0).

- Equation of the tangent plane to the level surface $f(x, y, z) = k$ at (x_0, y_0, z_0) is: $< f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) > \cdot < x - x_0, y - y_0, z - z_0 >= 0$, i.e., $f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$.

- Normal line to S at a point $P(x_0, y_0, z_0)$ is the line passing through P and perpendicular to the tangent plane:

  $$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}.$$

Example. Suppose that the temperature of a room at a point (x, y, z) is given by

$$T(x, y, z,) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}\ C^o.$$

(1) In which direction does the temperature increase fastest at the point $(2,1,1)$?
(2) What is the maximum rate of increase?
Solution: (1)

$$\nabla T(x,y,z) = T_x i + T_y j T_z k = \frac{160}{(1+x^2+2y^2+3z^2)^2}(-xi - 2yj - 3zk),$$

$$\nabla T(2,1,1) = \frac{8}{5}(-2i - 2j - 3k).$$

(2)

$$|\nabla T(2,1,1)| = \frac{8\sqrt{17}}{5}.$$

Example. Find the equation of the tangent plane and the normal line at the point (2,1,9) to the ellipsoid $\frac{x^2}{12} + \frac{y^2}{3} + \frac{z^2}{27} = 1$.

Solution: Tangent plane: $x + 2y + 2z - 22 = 0$.

14.7. Maximum and Minimum Values

Definition 2. *We say that a function $f(x,y)$ has a relative (local) maximum at a point (x_0, y_0) if there is a circle centered at (x_0, y_0) such that*

$$f(x,y) \geq f(x_0, y_0)$$

for all (x,y) in that circle; $f(x,y)$ has a relative (local) minimum at a point (x_0, y_0) if there is a circle centered at (x_0, y_0) such that

$$f(x,y) \leq f(x_0, y_0)$$

for all (x,y) in that circle.

Definition 3. *The critical points of a function $f(x,y)$ are those points (x_0, y_0) for which $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$, or if $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ is undefined. Saddle point: The graph of the function crosses the tangent plane at this point.*

Example 58. *Find the critical point(s) of $f(x,y) = x^3 - 3xy + y^3$*

Solution: The first partial derivatives are $f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = -3x + 3y^2$. Setting $f_x = 0$ and $f_y = 0:$ $3x^2 - 3y = 0$, $-3x + 3y^2 = 0$. We imply that $(x,y) = (0,0), (1,1)$.
First-Partials Test for Relative Extrema: If f has a relative extrema at (a,b), and the first partial derivatives exist in a circle centered at (a,b), then (a,b) is a critical point.

Second-Partials Test for Relative Extrema: Assume that f has a continuous partial derivatives on an open region containing (a, b). Let (a, b) be a critical point of f. Denote

$$d(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 .$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a relative minimum.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a relative maximum.
3. If $d < 0$, then $f(a, b)$ has a saddle point.
4. If $d = 0$, then the second derivatives test gives nothing.

Example 59. *Find the critical point(s) of $z = (x + y)(xy + xy^2)$ and classify them.*

Solution:

$$\frac{\partial z}{\partial x} = y(2x + y)(y + 1), \quad \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x + 1) + x\right].$$

$$\frac{\partial z}{\partial x}(x) = 0, \Rightarrow y = 0, -1, -2x.$$

Set

$$\frac{\partial z}{\partial y} = 0.$$

- If $y = 0$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x + 1) + x\right] = x^2, \Rightarrow x = 0.$$

- If $y = -1$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x + 1) + x\right] = x(1 - x), \Rightarrow x = 0, 1.$$

- If $y = -2x$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3(-2x)^2 + 2(-2x)(x + 1) + x\right] = 4x^2(2x - 1), \Rightarrow x = 0, \frac{1}{2}.$$

So critical points are

$$(a, b) \in \{(0, 0), (0, -1), (1, -1), (\frac{1}{2}, -1)\}.$$

To test all of them, we use the Second-Partials Test.

$$d(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = 2y(y+1)(2(3y+x+1))x - (3y^2+y(4x+2)+2x)^2.$$

- $d(0,-1) = -1$, $d(1,-1) = -1$, so $(0,-1), (1,-1)$ are saddle points.
- $d(0,0) = 0$, $d(\frac{1}{2},-1) = 0$, the test fails.

Example 60. *Classify the critical points of* $f(x,y) = x^3 - 3xy + y^3$

Solution:

$$f_x(x,y) = 3x^2 - 3y, \quad f_y(x,y) = -3x + 3y^2, \Rightarrow$$

$$f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 6y, \quad f_{xy}(x,y) = -3.$$

Setting $f_x = 0$ and $f_y = 0 :$ $3x^2 - 3y = 0$, $-3x + 3y^2 = 0$. We imply that $(x,y) = (0,0), (1,1)$.

$$d(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 36xy - 9.$$

- $d(0,0) = -9$, $d(1,-1) = -1$, so $(0,0)$ is a saddle point.
- $d(1,1) = 27$, $f_{xx}(1,1) = 6 > 0$, so $f(1,1)$ is a relative minimum.

Theorem 1. *A continuous function defined on a bounded region D attains a global maximum and a global minimum in D.*

Method. Find the value of the function at all critical points in D, and find local extreme values on the boundary of D, and compare.

Example 61. *Classify the critical points of* $f(x,y) = x^3 - 3xy + y^3$, $0 \leq x \leq 2, 0 \leq y \leq 2$. *Find the absolute max and min.*

Solution: Inside: we see from the example above that $(0,0)$ is a saddle point, $f(1,1)$ is a relative minimum.

On $x = 0, 0 \leq y \leq 2$, we have $z = y^3$. $max1 = 8$ when $(y = 2), min1 = 0$ when $(y = 0)$.

On $x = 2, 0 \leq y \leq 2$, we have $z = 8 + y^3 - 6y$. $z' = 3y^2 - 6 = 0 \Rightarrow y = \sqrt{2}$. $max2 = 8$ when $(y = 0), min2 = 8 - 4\sqrt{2}$ when $(y = \sqrt{2})$.

On $y = 0, 0 \leq x \leq 2$, we have $z = x^3$. $max1 = 8$ when $(x = 2), min1 = 0$ when $(x = 0)$.

On $y = 2, 0 \leq x \leq 2$, we have $z = x^3 + 8 - 6x$. Similarly $max4 = 8$ when $(x = 2), min4 = 8 - 4\sqrt{2}$ when $(x = \sqrt{2})$.

The global (absolute) maximum is 8 attained at x = 0, y = 2, or x = 2, y = 0. The global (absolute) minimum is 1 attained at (0, 0).

14.8. Lagrange Multipliers

Case 1. Two variables with one constraint: Find the extreme values of the function $z = f(x, y)$ subject to constraint $g(x, y) = 0$.

Interpretation. Find extreme values of f (x, y) on a curve.

When z attains an extreme value a, the level curve $f(x, y) = a$ and $g(x, y) = 0$ have the same tangent line. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant.

Method of Lagrange Multipliers: Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0.$$

For each solution (x, y, λ) of this system of equations, find the value of $f(x, y)$. The maximum is the maximum value of z, and the minimum is the minimum of z.

Example 62. *$z = f(x, y) = xy$, subject to $x^2 + y^2 = 1$. Find the maximum and minimum.*

Solution. Here $g(x, y) = x^2 + y^2 - 1$. $\nabla f =< y, x >, \nabla g =< 2x, 2y >$. So we have

$$\begin{aligned} y &= \lambda 2x, \\ x &= \lambda 2y, \\ x^2 + y^2 &= 1. \end{aligned}$$

We have $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$.

The maximum value of z is $z(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1/2$,

and the minimum value of z is $z(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = z(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -1/2$.

Case 2. Three variables with one constraint: Find the extreme values of the function $w = f(x, y, z)$ subject to constraint $g(x, y, z) = 0$.

Interpretation. Find extreme values of f (x, y,z) on a surface.

When w attains an extreme value a, the level curve $f(x, y, z) = a$ and $g(x, y, z) = 0$ have the same tangent plane. Hence their gradient vectors have the same or opposite direction: $\nabla f = \lambda \nabla g$, λ is a constant.

Method of Lagrange Multipliers: Solve the system of equations:

$$\nabla f = \lambda \nabla g, \quad g(x, y, z) = 0.$$

For each solution (x, y, z, λ) of this system of equations, find the value of $f(x, y, z)$. The maximum is the maximum value of w, and the minimum is the minimum of w.

Example 63. $w = f(x, y, z) = xyz, 2xz + 2yz + xy - 12 = 0$. *Find the maximum and minimum.*

Solution. Here $g(x, y, z) = 2xz + 2yz + xy - 12$. $\nabla f =< yz, xz, xy >, \nabla g =< 2z + y, 2z + x, 2x + 2y >$. So we have

$$\begin{aligned} yz &= \lambda(2z + y), \\ xz &= \lambda(2z + x), \\ xy &= \lambda(2x + 2y), \\ 2xz + 2yz + xy - 12 &= 0. \end{aligned}$$

We have $(x, y, z) = (2, 2, 1), (-2, -2, -1)$.

The maximum value of w is $w = f(2, 2, 1) = 4$;

and the minimum value of w is $w = f(-2, 2, -1) = -4$.

Case 3. Three variables with two constraints: Find the extreme values of the function $w = f(x, y, z)$ subject to constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Interpretation. Find extreme values of f (x, y,z) on a 3-D curve.

When w attains an extreme value a, the tangent lines of the curve $g(x, y, z) = 0, h(x, y, z) = 0$, are in the tangent plane of the level surface $f(x, y, z) = a$. Hence, the gradient vector of level surface $(x, y, z) = a$ and the gradient vectors of $g(x, y, z) = 0, h(x, y, z) = 0$ are in the same plane:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

λ and μ are a constants.

Method of Lagrange Multipliers: Solve the system of equations:

$$\nabla f = \lambda \nabla g + \mu \nabla h, \quad g(x, y, z) = 0, h(x, y, z) = 0.$$

For each solution (x, y, z, λ, μ) of this system of equations, find the value of $f(x, y, z)$. The maximum is the maximum value of w, and the minimum is the minimum of w.
Remark. λ, μ are called Lagrange Multipliers.

Example 64. $w = f(x, y, z) = x + y + 7z$ *subject to* $x - y + z = 1, x^2 + y^2 = 1$. *Find the maximum and minimum.*

Solution. Here $g(x,y,z) = x - y + z - 1$, $h(x,y,z) = x^2 + y^2 - 1$,, $\nabla f =< 1, 1, 7 >$, $\nabla g =< 1, -1, 1 >$, $\nabla h =< 2x, 2y, 0 >$. So we have

$$\begin{aligned} 1 &= \lambda + \mu 2x, \\ 1 &= -\lambda + \mu 2y, \\ 7 &= \lambda, \\ x - y + z - 1 &= 0, \\ x^2 + y^2 - 1 &= 0. \end{aligned}$$

We have $(x, y, z) = (-0.6, 0.8, 2.4), (0.6, -0.8, -0.4)$.

The maximum value of w is $w = f(-0.6, 0.8, 2.4) = 17$;

and the minimum value of w is $w = f(0.6, -0.8, -0.4) = -3$.

Chapter 15: Multiple Integrals

15.1. Double Integrals over Rectangles

Consider the function $z = f(x, y)$ defined on a rectangle $R : a \leq x \leq b, c \leq y \leq d$. Subdivide $[a, b]$ into $a = x_0 < x_1 < \cdots < x_m = b$, and $[c, d]$ into $c = y_0 < y_1 < \cdots < y_n = d$.

The double integral over this rectangle is

$$\iint\limits_R f(x, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A,$$

where $\Delta A = \Delta x \Delta y$, $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, $x_{i-1} \leq x_{ij} \leq x_i$, $y_{i-1} \leq y_{ij} \leq y_i$.

Geometric meaning: If $f(x, y) \geq 0$, then it is the volume of the solid under the graph of $f(x, y)$, above the x-y plane, bounded by R.

The average value of the function defined inside R is

$$f_{ave} = \frac{1}{(b-a)(d-c)} \iint\limits_R f(x, y)dA.$$

Numerical Approximation: Midpoint Rule: To approximate a double integral numerically, we may choose the middle point in each small rectangle as the sample point, or the top right corner (x_i, y_j) as the sample point.

15.2. Iterated Integrals

Fubini's Theorem: If $f(x, y)$ is continuous on the rectangle $R : a \leq x \leq b, c \leq y \leq d$, then

$$\iint\limits_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dy\, dx = \int_c^d \int_a^b f(x, y)dx\, dy.$$

The right hand side is called iterated integral. By this theorem, we can evaluate a double integral using an iterated integral.

Special case: If $f(x, y) = g(x)h(y)$, then the iterated integral becomes the product of two integrals.

$$\iint\limits_R f(x, y)dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dx\right).$$

Example 65. $z = y\sin(xy)$, $R = \{(x, y)|1 \leq x \leq 2, 0 \leq y \leq \pi/2\}$.

Solution:

$$\iint_R f(x,y)dA = \int_0^{\pi/2}\int_1^2 y\sin(xy)dx\,dy = \int_0^{\pi/2}(-\cos(xy))\,|_1^2 dy$$

$$= \int_0^{\pi/2}(-\cos(2y)+\cos(y))\,dy = \left(-\frac{1}{2}\sin(2y)+\sin(y)\right)|_0^{\pi/2} = 1.$$

Remark. We may use the other order to integrate with respect to x first, but it involves an integral that is harder to evaluate.

Example 66. $z = 16 - x^2 - 2y^2,\ R = [0,2]\times[0,2]$.

Solution:

$$\iint_R zdA = \int_0^2\int_0^2 (16-x^2-2y^2)dx\,dy = \int_0^2\left(16x - \frac{1}{3}x^3 - 2xy^2\right)|_{x=1}^2 dy$$

$$= \int_0^2\left(\frac{88}{3} - 4y^2\right)dy = 48.$$

Example 67. *Find the volume of the solid bounded by $z = \sin x\cos y$, the planes $x = \pi/2$ and $y = \pi/2$, and the three coordinates.*

Solution:

$$V = \iint_R zdA = \int_0^{\pi/2}\int_0^{\pi/2}\sin x\cos y dx\,dy = 1.$$

15.3. Double Integrals over General Regions

Let D be a general finite region in the plane, and $z = f(x,y)$ be a 2-varaible function. Define

$$\tilde{f}(x,y)\begin{cases} f(x,y), & (x,y)\in D; \\ 0, & (x,y,z)\notin D. \end{cases}$$

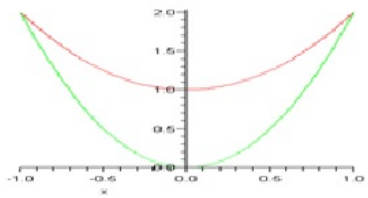
Let R be a rectangle that contains D. Then

$$\iint_D f(x,y)dA = \iint_R \tilde{f}(x,y)dA.$$

Type I: A region R is of Type I, if $R = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$. Then

$$\iint_R f(x,y)dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx.$$

Example 68. *Let $z = x + 2y$, R be the region bounded by $y = 2x^2$ and $y = x^2 + 1$.*



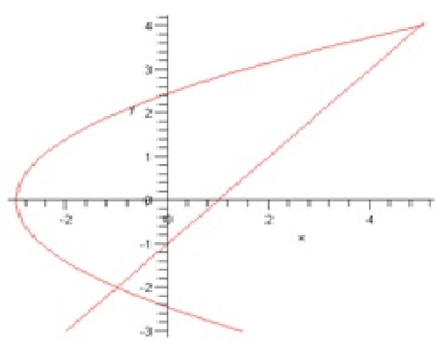
Solution: The intersection points of $y = 2x^2$ and $y = x^2 + 1$ are $x = \pm 1$. Thus $R = \{(x, y) : -1 \le x \le 1, 2x^2 \le y \le x^2 + 1\}$.

$$\iint_R z dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x + 2y) dy\, dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + 1)\, dx = \frac{32}{15}.$$

Type II: A region R is of Type II, if $R = \{(x, y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$. Then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx\, dy.$$

Example 69. *Let $z = xy$, R be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.*



Solution: The intersections of $y = x - 1$ and $y^2 = 2x + 6$ are $y = -2, 4$. Thus $R = \{(x, y) : -2 \le y \le 4, \frac{1}{2}y^2 - 3 \le x \le y + 1\}$.

$$\iint_R f(x, y) dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy dx\, dy = 36.$$

Example 70. *Evaluate the volume of the tetrahedron bounded by planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.*

Solution: The vertices of this tetrahedron are (0, 0, 0), (0, 1, 0), (1, 1 / 2, 0), and (0, 0, 2). Then $z = 2 - x - 2y$, $R = \{(x, y) : 0 \le x \le 1, x/2 \le y \le 1 - x/2\}$.

$$V = \iint_R f(x, y) dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dy\, dx = \frac{1}{3}.$$

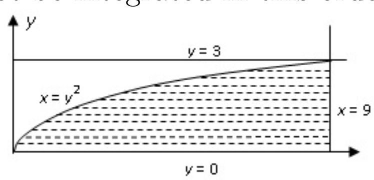
General Region:

A region R has to be subdivided into a number of such regions and calculate the integral separately. If a region can be regarded either as a Type I region, or a Type II region, in some cases, the order of integration is significant.

Changing the order of integration: Some regions can be regarded as of Type I or of Type II. We may use two different ways to express a double integral over such a region as iterated integral. In some cases, both ways are appropriate, and give the same result. However, in some cases, one iterated integral can be evaluated, but the other cannot.

Example 71. *Find* $\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy$.

Since the integral $\int_{y^2}^9 y\sin(x^2)dx$ cannot be integrated analytically, this iterated integral cannot be integrated in this order. We need to change the order.



$$\int_0^3 \int_{y^2}^9 y\sin(x^2)dxdy = \iint_R y\sin(x^2)dA = \int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)dydx = \frac{1-\cos 81}{4}.$$

Properties of Double Integrals:

(i) $\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$.

(ii) $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$.

(iii) If $f(x,y) \le g(x,y)$ for all $(x,y) \in R$, then $\iint_R f(x,y)dA \le \iint_R g(x,y)dA$.

(iv) If $R = R_1 \bigcup R_2, R_1 \bigcap R_2 = \emptyset$, then $\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$.

(v) $\iint_R dA =$ the area of R.

(vi) If $m \le f(x,y) \le M$ for all $(x,y) \in R$, then $mA \le \iint_R f(x,y)dA \le MA$, where A is the area of R.

15.4. Double Integrals in Polar Coordinates

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. A polar rectangle is defined to be $R = \{(r,\theta) : \alpha \le \theta \le \beta, a \le r \le b\}$, where $0 \le \beta - \alpha \le 2\pi$.

The double integral over a polar rectangle R is evaluated by the iterated integral:

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta, \quad dA = r dr d\theta.$$

Example 72. *Find* $\iint\limits_R z dA$*, where* $z = e^{-(x^2+y^2)}$*,* $R = \{(r,\theta) : 0 \le \theta \le \pi/2, 0 \le r \le a\}$*.*

$$\iint\limits_R z dA = \int_0^{\pi/2} \int_0^a e^{-(r^2)} r dr d\theta = \frac{(1-e^{-a^2})\pi}{4}.$$

If $R = \{(r,\theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Example 73. *Find the volume of the solid above the x-y plane, under the paraboloid* $z = x^2 + y^2$*, and inside the cylinder* $x^2 + y^2 - 2x = 0$*.*

The base of the cylinder is the circle in x-y plane $(x-1)^2 + y^2 = 1$ with center (1, 0) and radius 1. In polar coordinate system, the equation of the circle is $r = 2\cos\theta$. Thus

$$R = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}.$$

$$V = \iint\limits_R z dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos(4\theta)\right) d\theta = \frac{3\pi}{2}.$$

15.7. Triple Integrals

15.7.1 Triple Integral over a Rectangular Box

A rectangular box is the region in 3-dimensional space defined by

$$B = \{(x,y,z) : a \le x \le b, c \le y \le d, r \le z \le s\}.$$

Fubini's Theorem: If $f(x,y,z)$ is continuous on the rectangular box B, then

$$\iiint\limits_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z) dx\, dy\, dz.$$

This integral can also be evaluated by the other orders of the variables.

Example 74. *Find* $\iiint\limits_B (x+y+z)dV$*, where* $B = \{(x,y,z) : 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\}$.

$$\iiint\limits_B (x+y+z)dV = \int_0^2 \int_0^2 \int_0^2 (x+y+z)dx\,dy\,dz = \int_0^2 \int_0^2 (\frac{1}{2}x^2 + xy + xz)\Big|_{x=0}^2 dy\,dz$$

$$= \int_0^2 \int_0^2 (2y+2z+2)\,dy\,dz = \int_0^2 (y^2+2yz+2y)\big|_{y=0}^2\,dz = \int_0^2 (4z+8)\,dz = 24.$$

15.7.2 Triple Integrals over a General Region

Let E be a general finite region in 3-dimensional space, and $w = f(x,y,z)$ be a 3-varaible function. Define

$$\tilde{f}(x,y,z) \begin{cases} f(x,y,z), & (x,y,z) \in E; \\ 0, & (x,y,z) \notin E. \end{cases}$$

Let B be a rectangular box that contains E. Then

$$\iiint\limits_E f(x,y,z)dV = \iiint\limits_B \tilde{f}(x,y,z)dV.$$

Type I. The region is bounded by a cylinder $F(x,y) = 0$, and the graphs of two functions of x and y:

$$E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\},$$

where D is the region in (x,y) plane bounded by the graph of $F(x,y) = 0$. A triple integral over a region E of type I is evaluated by

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right) dA.$$

Type II. The region is bounded by a cylinder $F(y,z) = 0$, and the graphs of two functions of y and z:

$$E = \{(x,y,z) : (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\},$$

where D is the region in (y,z) plane bounded by the graph of $F(y,z) = 0$. A triple integral over a region E of type II is evaluated by

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx \right) dA.$$

Type III. The region is bounded by a cylinder $F(x,z)=0$, and the graphs of two functions of x and z:

$$E=\{(x,y,z):(x,z)\in D, u_1(x,z)\le y\le u_2(x,z)\},$$

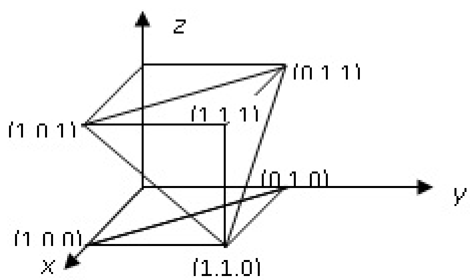
where D is the region in (x,z) plane bounded by the graph of $F(x,z)=0$. A triple integral over a region E of type III is evaluated by

$$\iiint_E f(x,y,z)dV=\iint_D\left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right)dA.$$

Remark. If a region is not of any of these types, we can subdivide this region into a finite number of regions of these types. The triple integral is the sum of triple integrals over sub-regions.

Example 75. *Evaluate* $\iiint_E (x+y)dV$ *, where* E *is the tetrahedron bounded by planes* $x=1, y=1, z=1,$ *and* $x+y+z=2$.

Solution: The solid is shown in the following diagram:

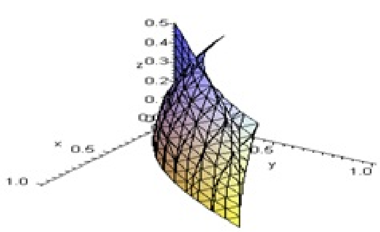


Solution: Let D be the triangle in xy- plane bounded by $x=1, y=1, x+y=1$. Hence

$$\iiint_E (x+y)dV=\iint_D\left(\int_{2-(x+y)}^{1}(x+y)dz\right)dA$$

$$=\int_0^1\int_{1-x}^1\int_{2-(x+y)}^1 (x+y)dz\,dy\,dx=\frac{1}{4}.$$

Example 76. *Evaluate* $\iiint_E zdV$ *, where* E *is the region bounded by paraboloid* $y=x^2+z^2$ *and the two planes* $x=y, z=0$.



Solution: Let D be the region bounded by $y = x^2$ and $y = x$. The intersections of the two curves are $(0,0), (1,1)$. Hence

$$\iiint\limits_E z dV = \iint\limits_D \left(\int_0^{\sqrt{y-x^2}} z dz \right) dA$$

$$= \int_0^1 \int_{x^2}^x \int_0^{\sqrt{y-x^2}} z dz\, dy\, dx = \int_0^1 \int_{x^2}^x \frac{1}{2}(y - x^2)\, dy\, dx = \frac{1}{120}.$$

Remark. If a solid E is bounded by two planes $z = a$, and $z = b$. The cross section at a given value z is D_z. Then the triple integral over E can also be evaluated by

$$\iiint\limits_E f(x,y,z) dV = \int_a^b \iint\limits_{D_z} f(x,y,z) dA\, dz.$$

15.8. Triple Integrals in Cylindrical Coordinates

Cylindrical coordinate system uses (r, θ, z) to specify a point in space, where r and θ are polar coordinates of the projection of the point on the xy-plane.

$$x = r\cos\theta, y = r\sin\theta, z = z; \quad r = \sqrt{x^2 + y^2}, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}, z = z.$$

Example 77. $(r, \theta, z) = (-1, \pi, 2) \Rightarrow (x, y, z) = (1, 0, -2)$; $(x, y, z) = (\sqrt{3}, -1, 2) \Rightarrow (r, \theta, z) = (2, -\pi/6, 2)$.

Suppose the region E of integration is of type I. A triple integral can be evaluated with the cylindrical coordinates:

$$x = r\cos\theta, y = r\sin\theta, z = z, dV = r dz dr d\theta.$$

Let D be the projection of E onto the xy-plane. Then

$$\iiint\limits_E f(x,y,z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

The other orders of integration are also possible depending on the definition of the region. For example, we may integrate with respect to θ and r first, and then z.

Let D_z be the intersection of region E and a plane parallel to the xy-plane with z being a constant. Then the triple integral can be evaluated by integrating for the region D_z first and integrate the result with respect to z.

$$\iiint\limits_E f(x,y,z) dV = \int_a^b \iint\limits_{D_z} f(r\cos\theta, r\sin\theta, z) r dr d\theta\, dz.$$

Example 78. *Find the integral* $\iiint\limits_E zdV$ *, where* E *is the region under the unit sphere and above the plane* $z = 1/2$.

Solution: In cylindrical coordinates, the unit sphere is given by $r^2 + z^2 = 1$. The projection of the intersection of the unit sphere and the plane $z = 1/2$ onto xy-plane is the region $x^2 + y^2 \leq 1 - z^2 = 1 - 1/4$, or $x^2 + y^2 \leq 3/4$. In cylindrical coordinates, we obtain $0 \leq r \leq \sqrt{3}/2$, $0 \leq \theta \leq 2\pi$, $1/2 \leq z \leq \sqrt{1 - r^2}$. Thus

$$\iiint\limits_E zdV = \int_0^{2\pi} \int_0^{\sqrt{3}/2} \int_{1/2}^{\sqrt{1-r^2}} zrdzdrd\theta = \frac{9\pi}{64}.$$

Remark. This integral can also be evaluated in another order: Let D be $x^2 + y^2 \leq 3/4$.

$$\iiint\limits_E zdV = \int_{1/2}^1 \iint_D zdAdz = \int_{1/2}^1 \int_0^{2\pi} \int_0^{\sqrt{1-z^2}} zrdrd\theta dz.$$

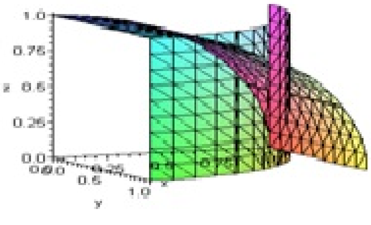
Example 79. *Find the integral* $\iiint\limits_E (x^2 + y^2)dV$ *, where* E *is the solid bounded by*

$$-2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, \sqrt{x^2 + y^2} \leq z \leq 2.$$

Solution: Note that E is the region bounded by the cone $z = x^2 + y^2$ and the plane $z = 2$. In cylindrical coordinates, we obtain $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, $r^2 \leq z \leq 2$. Thus

$$\iiint\limits_E (x^2 + y^2)dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^2 r^2 rdzdrd\theta = \frac{16\pi}{5}.$$

Example 80. *Find the integral* $\iiint\limits_E zdV$ *, where* E *is the solid bounded by* $x^2 + y^2 = 1, x^2 + z^2 = 1$.



Solution. We look at the part B of the solid E in the first octant. We have

$$B: \ 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq z \leq \sqrt{1 - x^2} = \sqrt{1 - r^2\cos^2\theta}.$$

Thus

$$\iiint\limits_B zdV = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} zrdzdrd\theta$$

$$= \int_0^{\pi/2} \frac{1}{8}(2-\cos^2\theta)d\theta = \int_0^{\pi/2} \frac{1}{16}(3-\cos 2\theta)d\theta = \frac{3\pi}{32}.$$

$$\iiint\limits_E zdV = 8\iiint\limits_B zdV = \frac{3\pi}{4}.$$

Example 81. *Find the volume of the solid bounded by* $x^2+y^2=1, x^2+z^2=1.$

Solution. We look at the part B of the solid E in the first octant. We have

$$B:\ 0\le r\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le z\le\sqrt{1-x^2}=\sqrt{1-r^2\cos^2\theta}.$$

Thus

$$\iiint\limits_B dV = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2\cos^2\theta}} rdzdrd\theta$$

$$= \int_0^{\pi/2}\int_0^1 r\sqrt{1-r^2\cos^2\theta}drd\theta = \int_0^{\pi/2}\frac{1-\sin^3\theta}{3\cos^2\theta}d\theta$$

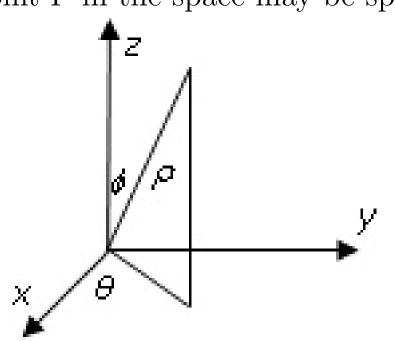
$$= \frac{1}{3}\int_0^{\pi/2}\left(\frac{1-\sin\theta+\sin\theta\cos^2\theta}{\cos^2\theta}\right)d\theta = \frac{1}{3}\int_0^{\pi/2}\left(\frac{1}{\cos^2\theta}-\frac{\sin\theta}{\cos^2\theta}+\sin\theta\right)d\theta$$

$$= \left(\tan\theta-\frac{1}{\cos\theta}-\cos\theta\right)\Big|_0^{\pi/2} = \left(\frac{\sin\theta-1}{\cos\theta}-\cos\theta\right)\Big|_0^{\pi/2} = \frac{2}{3},\quad \lim_{\theta\to\pi/2}\frac{\sin\theta-1}{\cos\theta}=0.$$

$$\iiint\limits_E dV = 8\iiint\limits_B dV = \frac{16}{3}.$$

15.9. Triple Integrals in Spherical Coordinates

A point P in the space may be specified (ρ,θ,ϕ):



$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi, dV = \rho^2 \sin\phi d\rho d\phi d\theta,$$

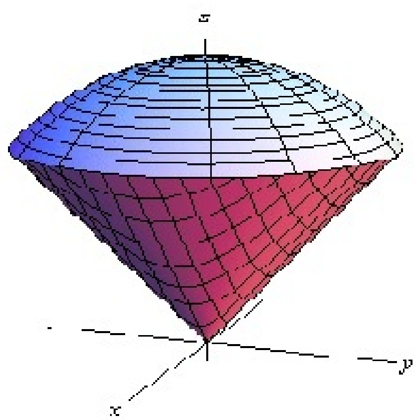
where $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2 sin\phi$.

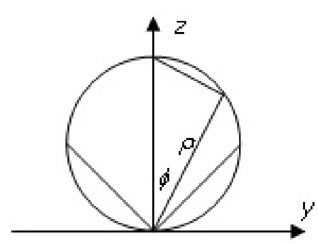
If a region E is specified in spherical coordinates, then

$$\iiint\limits_E f(x,y,z)dV = \iiint\limits_E f(\rho sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\phi d\theta,$$

where the order of integration depends on the definition of E.

Example 82. *Find the volume of the solid bounded by the cone $x^2 + y^2 - z^2 = 0$ and the upper half of the sphere $x^2 + y^2 + (z - 1/2)^2 = 1/4$.*





Solution: The intersection of this solid and the yz-plane is shown above. We need to find ranges for ρ, θ, ϕ. Note that the solid is within the cone and below the sphere, we have

$$x^2 + y^2 - z^2 \le 0, \Rightarrow \rho^2 \sin^2\phi - \rho^2\cos^2\phi \le 0, \Rightarrow \sin\phi \le \cos\phi, \Rightarrow 0 \le \phi \le \frac{\pi}{4}.$$

$$x^2 + y^2 + (z-1/2)^2 \le 1/4 \Rightarrow x^2 + y^2 + z^2 - z \le 0 \Rightarrow \rho^2 - \rho\cos\phi \le 0 \Rightarrow 0 \le \rho \le \cos\phi.$$

The intersections of $x^2 + y^2 - z^2 = 0$ and $x^2 + y^2 + (z - 1/2)^2 = 1/4$ are $z = 0, 1/2$, which gives the domain in the xy-plane: $x^2 + y^2 = 1/4$. Thus $0 \le \theta \le 2\pi$.

$$V = \iiint\limits_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\phi \sin\phi d\phi d\theta = \frac{\pi}{8}.$$

Example 83. *Find the integral $I = \iiint\limits_E \sqrt{x^2 + y^2 + z^2} dV$, where E is region between the sphere $x^2 + y^2 + z^2 = 1$, and the sphere $x^2 + y^2 + z^2 = 4$.*

Solution: It is easy to see that $1 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Thus

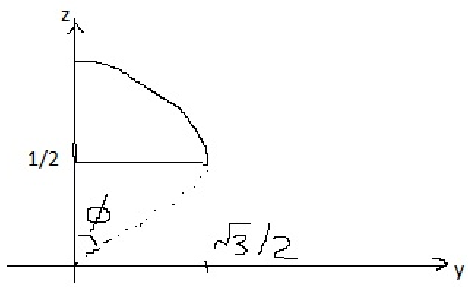
$$I = \iiint\limits_E \sqrt{x^2 + y^2 + z^2} dV = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \sqrt{\rho^2} \rho^2 \sin \phi d\rho d\phi d\theta = 15\pi.$$

Example 84. *Find the integral* $I = \iiint\limits_E z dV$*, where* E *is region below the sphere* $x^2 + y^2 + z^2 = 1$ *and above the plane* $z = 1/2$.

Solution: The intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $z = 1/2$ is

$$x^2 + y^2 = 3/4, \Rightarrow 0 \le \theta \le 2\pi.$$

To find the interval for ϕ, we look at the intersection of the solid and the yz-plane:



$$y^2 = 3/4 \Rightarrow y = \sqrt{3}/2 \Rightarrow \tan \phi = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \Rightarrow \phi = \pi/3 \Rightarrow 0 \le \phi \le \pi/3.$$

$$z \ge 1/2 \Rightarrow \rho \cos \phi \ge 1/2 \Rightarrow \rho \ge \frac{1}{2 \cos \phi}.$$

$$x^2 + y^2 + z^2 \le 1 \Rightarrow \rho^2 \le 1 \Rightarrow \rho \le 1.$$

Thus

$$I = \iiint\limits_E z dV = \int_0^{2\pi} \int_0^{\pi/3} \int_{1/(2\cos\phi)}^1 \rho \cos \phi \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \frac{1}{64} \int_0^{2\pi} \int_0^{\pi/3} \sin \phi \cos \phi \left(16 - \frac{1}{\cos^4 \phi}\right) d\phi d\theta = \frac{9\pi}{64}.$$

Chapter 16: Vector Calculus

16.1. Vector Fields

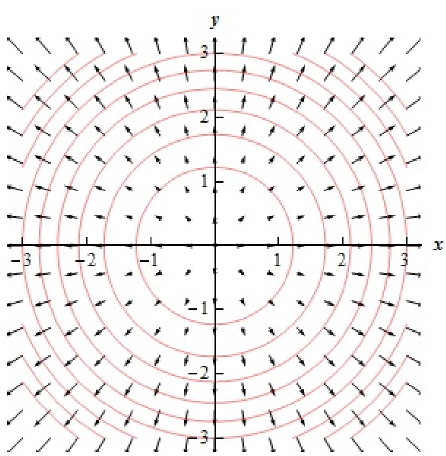
Vector Fields

In a two-dimensional space,vector function $\vec{F}(x,y) =< P(x,y), Q(x,y) >= P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a 2-dimensional vector field.

In a three-dimensional space, $\vec{F}(x,y,z) =< P(x,y), Q(x,y), R(x,y) >= P(x,y)i + Q(x,y)\vec{j} + R(x,y)\vec{k}$ is a 3-dimensional vector field.

Vector fields can be visualized by diagrams.

Example 85. *Gradient Vector Fields: Let* $f(x,y) = x^2 + y^2$*. Then* $\vec{F}(x,y) = \nabla f(x,y) =< x, y >$ *is a vector field.*



Conservative Vector Fields

A vector field $\vec{F}$ is conservative if there exists a function f such that $\vec{F} = \nabla f$. In other words, a vector field is conservative if it is the gradient field of a (scalar) function. The function f is called the **potential function** of $\vec{F}$.

Remark. The potential function of a conservative vector field is not unique.

Example 86. $\vec{F} =< y\cos x, \sin x >$ *is a conservative vector field with potential function* $f(x,y) = y\sin x$.

$\vec{F} =< y2, 2xy + e3z, 3ye3z >$ *is a conservative vector field with potential function* $f(x,y,z) = xy^2 + ye^{3z}$.

16.2. Line Integrals

1. Line Integrals of Scalar fields in 2-D

Let C be a smooth curve given by $x = x(t), y = y(t), a \le t \le b$, or equivalently, by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$. Let the density at a point (x, y) on C be $f(x, y)$. Subdivide this curve segment into a number of small segments. The weight of a small segment of the curve is approximately $f(x^*, y^*)\Delta s$, where (x^*, y^*) is a point in this segment, and Δs is the length of this small segment. The sum $\sum f(x^*, y^*)\Delta s$ is an approximation of the total weight of the curve segment. The total weight of C is $\lim_{\Delta s \to 0} \sum f(x^*, y^*)\Delta s$.

Definition 4. *If f is defined on a smooth curve C, then the line integral of f along C is*

$$\int_C f(x, y)ds = \lim_{\Delta s \to 0} \sum f(x^*, y^*)\Delta s$$

if the limit exists.

This is also called the line integral of type I.

An interpretation of line integral: The area of a "fence" with C as the base, and the height is given by $f(x, y)$.

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x = x(t), y = y(t), a \le t \le b$, then

$$\int_C f(x, y)ds = \int_a^b f(x, y)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt, \quad ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

If C can be subdivide into a finite number of segments, and smooth on each segment, then the line integral is calculated for each segment and the sum is the line integral of C.

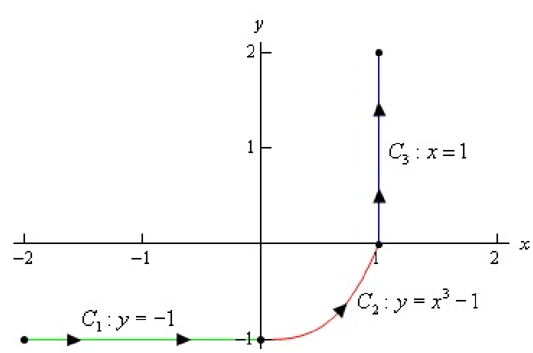
Example 87. *Find $\int_C yds$, where C is the cycloid $x = t - \sin t, y = 1 - \cos t$, $0 \le t \le 2\pi$.*

Solution:

$$\int_C yds = \int_0^{2\pi} y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}dt$$

$$= \int_0^{2\pi} 4\sin^3(\frac{t}{2})dt = -8\left[\cos(t/2) - \frac{1}{3}\cos^3(t/2)\right]_0^{2\pi} = \frac{32}{3}.$$

Example 88. *Evaluate $\int_C 4x^3 ds$, where C is the curve shown below:*



Solution: The three curves are:

$C_1 : x = t, y = -1, -2 \le t \le 0; \quad C_2 : x = t, y = t^3-1, 0 \le t \le 1; \quad C_3 : x = 1, y = t, 0 \le t \le 2.$

$$\int_{C_1} 4x^3 ds = \int_{-2}^{0} 4t^3\sqrt{1^2+0^2}dt = -16,$$

$$\int_{C_2} 4x^3 ds = \int_{0}^{1} 4t^3\sqrt{1^2+(3t^2)^2}dt = \frac{2}{27}(10^{3/2}-1),$$

$$\int_{C_3} 4x^3 ds = \int_{0}^{2} 4(1)^3\sqrt{0^2+1^2}dt = 8,$$

thus

$$\int_{C} 4x^3 ds = \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds = \frac{2}{27}(10^{3/2}-1) - 8.$$

Special cases: $ds = dx, or ds = dy$: $\int_C f(x,y)dx$ and $\int_C f(x,y)dy$ are called respectively line integral of f along C with respect to x and y. Suppose C is defined by parametric equations $x = u(t), y = v(t)$, $a \le t \le b$, then

$$\int_C P(x,y)dx + Q(x,y)dy = \int_a^b [P(u,v)u'(t) + Q(u,v)v'(t)]dt.$$

Example 89. *Find* $I = \int_C (x+y)dx + (x-y)dy$ *, where C is a curve defined by* $x = e^t \sin t, y = e^t \cos t, 0 \le t \le \pi/2$.

Solution:

$$I = \int_C (x+y)dx + (x-y)dy = \int_0^{\pi/2} [(e^t \sin t + e^t \cos t)(e^t \sin t)' + (e^t \sin t - e^t \cos t)(e^t \cos t)']dt$$

$$= \int_0^{\pi/2} 2e^{2t}\sin(2t)dt = \int_0^{\pi} e^w \sin(w)dw = \frac{1}{2}(e^{\pi}+1).$$

2. Line Integrals of Scalar fields in 3-D

Calculation of a line integral: If the smooth curve C is defined by parametric equations $x = x(t), y = y(t), z = z(t), a \le t \le b$, then

$$\int_C f(x,y,z)ds = \int_a^b f(x,y,z)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt.$$

Example 90. *Find* $\int_C (xy - z)ds$*, where C is the cycloid* $x = t, y = t^2, z = \frac{2}{3}t^3$, $0 \le t \le 1$.

Solution:

$$\int_C (xy-z)ds = \int_0^1 (xy-z)\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}dt = \int_0^1 \frac{1}{3}t^3\sqrt{1+4t^2+4t^4}dt = \frac{7}{36}.$$

3. Line Integral of Vector fields

Definition 5. *Let $\vec{F}$ be a continuous vector field defined on a smooth curve $C : \vec{r} = \vec{r}(t), a \le t \le b$. Then the line integral of $\vec{F}$ along C is:*

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r})\cdot \vec{r}'(t)dt = \int_C \vec{F}\cdot \vec{T}ds,$$

which can be interpreted as the total work done by this force vector field when this object is moving from one end of C to the other end of C.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P(x,y)dx + Q(x,y)dy, \;\; if \;\; \vec{F} =< P(x,y), Q(x,y) >;$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz,$$

if$\vec{F} =< P(x,y,z), Q(x,y,z), R(x,y,z) >$.

Example 91. *Find $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} =< x,y,z >$ and C is: $x=\cos t, y=\sin t, z=\sin t$, $0 \le t \le \pi/2$.*

Solution:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C xdx + ydy + zdz = \int_0^{\pi/2} [(\cos t)(\cos t)' + (\sin t)(\sin t)' + (\sin t)(\sin t)']dt$$

$$= \int_0^{\pi/2} \sin t\cos t dt = \frac{1}{2}.$$

16.3. The Fundamental Theorem for Line Integrals

Fundamental Theorem for Line Integral: Let $\vec{F} = \nabla f$ be a conservative vector field, and C be a smooth curve defined by parametric equation $C : \vec{r} = \vec{r}(t), a \le t \le b$. Then

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Path Independence of Line Integrals: Let $\vec{F}$ be a vector field continuous in an open connected region. The following statements are equivalent:

(a) The line integral is independent of path.

(b) The line integral is zero along any closed curve.

(c) $\vec{F}$ is a conservative vector field.

A Necessary and Sufficient Condition for a Vector Field to be Conservative: Let $\vec{F} =< P, Q >$ be a vector field in a simply-connected region D. Suppose P and Q have continuous first-order derivative in D, then $\vec{F}$ is conservative if and only if

$$P_y = Q_x.$$

Example 92. *(i) Show that the vector field* $\vec{F} =< 3 + 2xy, x^2 - 3y^2 >$ *is conservative.*

(ii) Find a potential function of this field.

(iii) Find $\int_C (3+2xy)dx + (x^2 - 3y^2)dy$ *, where C is a curve defined by* $x = e^t \sin t, y = e^t \cos t, 0 \le t \le \pi$.

Solution : (i) Let $P = 3 + 2xy$, $Q = x^2 - 3y^2$. Since $P_y = 2x = Q_x$, $\vec{F}$ is conservative.

(ii) Let f(x,y) be a potential function. Then $F =< f_x, f_y >=< P, q >$,

$$f_x = 3 + 2xy, f_y = x^2 - 3y^2.$$

$$f_x = 3 + 2xy \Rightarrow f(x,y) = 3x + x^2y + g(y) \Rightarrow f_y = x^2 + g'(y) \Rightarrow$$

$$x^2 - 3y^2 = x^2 + g'(y), \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3.$$

Hence,

$$f(x,y) = 3x + x^2y - y^3.$$

(iii) To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 1$. When $t = \pi, x = 0, y = -e^\pi$. Hence

$$\int_C (3+2xy)dx+(x^2-3y^2)dy = \int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\pi))-f(\vec{r}(0)) = f(0,-e^\pi)-f(0,1) = e^{3\pi}+1.$$

Example 93. *(i) Find a potential function* $f(x,y,z)$ *of the vector field* $\vec{F} =< z, 2yz, x + y^2 >$.

(ii) Find $\int_C \vec{F} \cdot d\vec{r}$ *, where C is a curve defined by* $x = t, y = t^2, z = 2t, 0 \le t \le 1$.

Solution : (i) Let f(x,y,z) be a potential function. Then $F =< f_x, f_y, f_z >=< z, 2yz, x + y^2 >$,

$$f_x = z, f_y = 2yz, f_z = x + y^2.$$

$$f_x = z \Rightarrow f = xz + g(y,z) \Rightarrow f_y = g_y = 2yz \Rightarrow g(y,z) = y^2z + h(z), f = xz + y^2z + h(z) \Rightarrow$$

$$f_z = x + y^2 + h'(z) = x + y^2, \Rightarrow h'(z) = 0 \Rightarrow h(z) = constant.$$

Hence,

$$f = xz + y^2z.$$

(iii) To find the line integral, look at the starting and ending point of the curve. When $t = 0, x = 0, y = 0, z = 0$. When $t = 1, x = 1, y = 1, z = 2$. Hence

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1,2) - f(0,0,0) = 4.$$

16.4. Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Green's Theorem. Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. If P and Q are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Example 94. *Use Greens Theorem to evaluate* $\int_C xydx + x^2y^3dy$ *where* C *is the triangle with vertices ,(0,0), (1,0), (1,2) , with positive orientation.*

Solution:

$$\int_C xydx + x^2y^3dy = \int_D (2xy^3 - x)dxdy = \frac{2}{3}.$$