INDU 371 Stochastic Models in Industrial Engineering

Midterm Exam

Question 1 (20 points): An airline conducted a survey amongst a large population of passengers and found that generally five percent of people making reservations for a flight do not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold 50 passengers. Compute the exact probability that there will be a seat available for every passenger who shows up for the flight.

Solution:

P(not showing up | T) = 0.05

P(showing up | T) = 0.95

n = 52, p = 0.95, q = (1-p)

P(X =< 50) = 1 – (X > 50) = 1- ($\binom{52}{51} 0.95^{51}(1-0.95)$ - ($\binom{52}{52} 0.95^{52}(0)$
$1 - 0.19 - .069 = 0.741$

Question 2 (20 points): In a certain factory, machines I, II, and III are all producing springs of the same length. Of their production, machines I, II, and III produce 2%, 1%, and 3% defective springs respectively. Of the total production of the springs in the factory, machine I produces 35%, machine II produces 25%, and machine III produces 40%. A spring is selected and it is observed to be defective. Given that this selected spring is defective, what is the probability that

a) It was produced by machine I?
b) It was produced by machine II?
c) It was produced by machine III?

Solution:

P(D|I) = 0.02
P(D|II) = 0.01
P(D|III) = 0.03
P(I) = 0.35
P(II) = 0.25
P(III) = 0.40

a) $$\Pr[I \mid D] = \frac{\Pr[D \mid I]\Pr[I]}{\Pr[D \mid I]\Pr[I] + \Pr[D \mid II]\Pr[II] + \Pr[D \mid III]\Pr[III]} = 0.325$$

b) $$\Pr[II \mid D] = \frac{\Pr[D \mid II]\Pr[II]}{\Pr[D \mid I]\Pr[I] + \Pr[D \mid II]\Pr[II] + \Pr[D \mid III]\Pr[III]} = 0.1163$$

c) $$\Pr[III \mid D] = \frac{\Pr[D \mid III]\Pr[III]}{\Pr[D \mid I]\Pr[I] + \Pr[D \mid II]\Pr[II] + \Pr[D \mid III]\Pr[III]} = 0.558$$

Question 3 (30 points): A man earns his living by polishing the shoes of his customer in his small "one chair" shoe polishing shop. Observations indicate that in time required to polish one person's shoes there may be zero, one or two new arrivals to his shop with probability 0.3, 0.4, and 0.3 respectively. The shop has a fixed number of chairs for a total of five people (excluding the owner); that is to say there are four chairs for the customers waiting and one chair for the customer whose shoes are being polished. No customer enters the shop if all the chairs are occupied. Let X_n be the number of the people in the shop at the completion of the n^{th} customer's shoe polish (assume this n^{th} customer leaves the shop immediately after receiving his service). Assume the process is a Markov Chain with independent, identically distributed arrivals.

a) Find the one step transition probability matrix for this process.
b) Assuming the limiting probability exist, write the complete set of equations required to determine the long run proportion of the time for the possible number of people in the shop. Do not solve the equations.

Solution:

$$\mathbf{\Lambda} = \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.0 & 0.0 \\ 0.3 & 0.4 & 0.3 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.4 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.4 & 0.3 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.7 \end{bmatrix}$$

$$V = VP$$

$$V_0 + V_1 + V_2 + V_3 + V_4 = 1$$

$$V_0 = 0.3V_0 + 0.3\,V_1$$

$$V_1 = 0.4V_0 + 0.4\,V_1 + 0.3\,V_2$$

$$V_2 = 0.3V_0 + 0.3\,V_1 + 0.4V_2 + 0.3V_3$$

$$V_3 = 0.3\,V_2 + 0.4V_3 + 0.3V_4$$

$$V_4 = 0.3V_3 + 0.7V_4$$

<u>Question 4 (30 points):</u> Consider a simple weather model which can be described by Markov Chain. We have two states, namely Dry State (State 0) and Wet State (State I). From a given day to the next, the current weather condition either does not change (with probability ¾) or changes (with probability ¼). Assuming that when we first start to observe this process, it was a dry day.

a) Specify the initial probability state vector for this Markov Chain.
b) Specify the transition matrix for this Markov Chain.
c) What is the probability that we will observe a wet day three days after the day we have started?

<u>Solution</u>:

a) $\pi = (1,0)$

b) $\mathbf{P} = \begin{array}{c} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \begin{matrix} 0 \\ 1 \end{matrix}\left.\vphantom{\begin{matrix} 0 \\ 1 \end{matrix}}\right\} \end{array}$

c) $\mathbf{P}^3 = (1,0)\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} = {}^{7}\!/_{16}$

Problem 1.

The number of times that an individual contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 3$. A new vaccine was developed to reduce the value of λ to $\lambda = 2$ for 75% of those taking the vaccine while it has no effect on the other 25% people. If David took the vaccine and did not get cold for a year, what is the probability that the vaccine did have effect on David?

***Solution*.**

$$\Pr[has\,effect \mid no\ cold] = \frac{\Pr[no\ cold \mid has\,effect]\Pr[has\,effect]}{\Pr[no\ cold \mid effect]\Pr[has\,effect] + \Pr[no\ cold \mid no\ effect]\Pr[no\ effect]}$$

$$= \frac{\Pr[X = 0 \mid has\ effect](0.75)}{\Pr[X = 0 \mid has\ effect](0.75) + \Pr[X = 0 \mid no\ effect](0.25)}$$

$$= \frac{\Pr[X = 0 \mid \lambda = 2](0.75)}{\Pr[X = 0 \mid \lambda = 2](0.75) + \Pr[X = 0 \mid \lambda = 3](0.25)}$$

$$= \frac{\left.(\lambda t)^i \frac{e^{-\lambda t}}{i!}\right|_{i=0,\lambda t=2} \times 0.75}{\left.(\lambda t)^i \frac{e^{-\lambda t}}{i!}\right|_{i=0,\lambda t=2} \times (0.75) + \left.(\lambda t)^i \frac{e^{-\lambda t}}{i!}\right|_{i=0,\lambda t=3} \times 0.25} = \frac{e^{-2} \times 0.75}{e^{-2} \times 0.75 + e^{-3} \times \big| 0.25} = 0.8908$$

Problem 2.

A survey result shows that a student on the Dean’s Honor List (HR) of one year may fall off the list in the next year if his/her GPA goes below a certain point, to become a “Very Good (VG)” student, or a “Satisfactory (SA)” student, if the GPA goes down further. Similarly, “Very Good” or “Satisfactory” students may move to other categories as their GPA changes each year. We consider that the process for a student moving from one category to another is a Markov process and construct the following 1-step transition matrix based on the collected data and analysis. We also know that the current distribution of the students in the 3 categories is 5%, 55% and 40%.

$$\mathbf{P} = \begin{array}{ccc|l} \multicolumn{3}{c|}{\text{category}} & \\ \text{HR} & \text{VG} & \text{SA} & \\ \hline 0.45 & 0.35 & 0.20 & \text{HR} \\ 0.20 & 0.50 & 0.30 & \text{VG} \\ 0.10 & 0.45 & 0.45 & \text{SA} \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}\text{category}$$

Assuming that the process is stationary, determine:

1) probability that an HR student falls off the Honor List after 1 year (at the beginning of Year 2)
2) probability that a current HR student’s name is on the Honor List at the beginning of Year 3.
3) probability that an HR student keeps his/her name on the Honor List for 2 consecutive years
4) probability that within 2 years a VG student can become (and remain) an HR student

Solution

1) probability that an HR student falls off the Honor List after 1 year (at the beginning of Year 2)
   $P_1 = P_{12} + P_{13} = 0.35 + 0.20 = 0.55$
2) probability that a current HR student’s name is on the Honor List at the beginning of Year 3.
   $P_2 = P_{11}P_{11} + P_{12}P_{21} + P_{13}P_{31} = 0.45 \times 0.45 + 0.35 \times 0.20 + 0.20 \times 0.10 = 0.2925$
3) probability that an HR student keeps his/her name on the Honor List for 2 consecutive years
   $P_3 = P_{11}P_{11} = 0.45 \times 0.45 = 0.2025$
4) probability that within 2 years a VG student can become (and remain) an HR student
   $P_4 = P_{21}P_{11} + P_{22}P_{21} + P_{23}P_{31} = 0.20 \times 0.45 + 0.50 \times 0.20 + 0.30 \times 0.10 = 0.22$

Sample Final Exam Problems with Solutions and Formula Sheet

Sample Problem 1

The transition probabilities for a Markov chain with 6 states (0, 1, 2, 3, 4 and 5) are given below:

$$\mathbf{P} = \begin{bmatrix} 0 & 0.8 & 0 & 0.2 & 0 & 0 \\ 0 & 0.1 & 0.7 & 0.2 & 0 & 0 \\ 0.4 & 0 & 0 & 0.4 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \end{bmatrix}$$

1. Determine, with brief explanations, if the process is irreducible.
2. For each state, determine, with brief explanations, if they are recurrent, transient, absorbing and/or periodic.

Solution

1. It is not irreducible as states 3, 4 and 5 construct a closed communicating class.
2. States 0, 1 and 2 are transient; states 3,4 and 5 are recurrent and periodic.

Sample Problem 2.

The users of two brands of personal computers (PC1 and PC2) may shift to the other brands. Assume that this is a stationary Markov process and the transition probability matrix is:

$$\begin{array}{c|cc} & PC1 & PC2 \\ \hline PC1 & 0.9 & 0.1 \\ PC2 & 0.2 & 0.8 \end{array}$$

There are 3,000 PCs in use in the marketing area and the average life of a PC is 4 years. Selling each computer can make a profit of $200.

1. In the long run, how many PC1 computers can be sold each year?
2. An advertising firm guarantees to decrease from 10% to 5% the fraction of PC1 customers who switch to PC2 after a purchase. How much PC1 company may consider paying the advertising firm to do the job each year?

Solution

1. From $\mathbf{v} = \mathbf{vP}$, we have: $[v_1, v_2] = [v_1, v_2]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

Then:

$$v_1 = 0.9v_1 + 0.2v_2$$
$$v_2 = 0.1v_1 + 0.8v_2$$

Replacing the second equation with $v_1 + v_2 = 1$ we have

$$v_1 = 0.9v_1 + 0.2v_2$$
$$v_1 + v_2 = 1$$

We have $v_1 = 2/3, v_2 = 1/3$.

Then PC1 computer can be sold 2,000 units every 4 years or 500 units each year.

2. The current transition will lead to a total profit of $500 \times \$200=\$100{,}000$ per year for PC1. The new transition matrix after the advertising firm does the job should be:

$$\begin{bmatrix} 0.95 & 0.05 \\ 0.2 & 0.8 \end{bmatrix}$$

Solving

$$[v_1, v_2] = [v_1, v_2]\begin{bmatrix} 0.95 & 0.05 \\ 0.2 & 0.8 \end{bmatrix}$$

with $v_1 + v_2 = 1$, we have $v_1 = 0.8, v_2 = 0.2$. The number of PC1 computers can be sold every 4 years would be $3{,}000 \times 0.8 = 2{,}400$. Then each year it can sell 600 units. The extra profit would be $100 \times 200=\$20{,}000$. So PC1 may consider paying the firm up to $20,000 per year to do the commercials.

Problem 3

The outside temperature in spring time can be considered as a random process with continuous parameter time t. We consider only 3 temperature levels: Cold, Cool, and Warm. As we observe, the temperature may change from Cold to Cool, Cool to Warm and vice versa. If it is a Cold day, it will not bypass the Cool level to "jump" directly from Cold to Warm. Similarly, a Warm day will not bypass a Cool day to become a Cold day. Statistics show that the time for the temperature to be at the three different levels follows exponential distributions. Average Cold time is 5 days, average Cool time is 2 days and average Warm time is 3 days. If a Cool day changes, it is equally likely to change to either Cold or Warm.

1. Based on the given information, present the specific Chapman-Kolmogorov forward differential equations in matrix form. Do NOT solve these differential equations.
2. Estimate the numbers of Cold and Warm days in the 3 months (March, April and May) of the spring season.

Solution

1. $$\mathbf{P}'(t) = \begin{bmatrix} p'_{00}(t) & p'_{01}(t) & p'_{02}(t) \\ p'_{10}(t) & p'_{11}(t) & p'_{12}(t) \\ p'_{20}(t) & p'_{21}(t) & p'_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} -0.2 & 0.2 & 0 \\ 0.25 & -0.5 & 0.25 \\ 0 & 0.33 & -0.33 \end{bmatrix} = \mathbf{P}(t)\mathbf{\Lambda}, \mathbf{P}(0) = \mathbf{I}$$

2. Solving $\mathbf{q\Lambda} = 0$ **with** $q_0 + q_1 + q_2 = 1$ gives $q_0 = 0.417, q_1 = 0.333, q_2 = 0.250$. The number of cold days is $92 \times 0.417 = 38$ and the number of warm days is $92 \times 0.25 = 23$.

Sample Problem 4.

A one-man barber shop has a total of 10 seats. Inter-arrival times are exponential. An average of 20 prospective customers arrive each hour at the shop. Those customers who find the shop full do not enter. The barber takes an average of 12 minutes to cut each customer's hair. Service time is also exponential.

1. On the average, how many haircuts per hour will the barber complete?
2. On the average, how many customers are in the shop?

Solution

1. This is a *M/M/1/K* model with *K*=10. The shop can only take a maximum of 10 customers. Since the customers will leave if it is full, the number of customers actually enter will be $\lambda q_0 + \lambda q_1 + \lambda q_2 + \ldots + \lambda q_9 = \lambda(1 - q_{10})$ each hour. All customers entered the shop will have a haircut so the barber will give an average of $\lambda(1 - q_{10})$ haircuts per hour. As we are given that *K*=10, $\lambda = 20$ per hour and $\mu = 5$ per hour, we have $\rho = 20/5 = 4$. The *M/M/K* probability calculation shows that:

$$q_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-4}{1-4^{11}}$$

From

$$q_n = \rho^n \frac{1-\rho}{1-\rho^{K+1}}$$

We have:

$$q_{10} = \rho^{10} \frac{1-\rho}{1-\rho^{11}} = 4^{10} \frac{(-3)}{1-4^{11}} = 0.75$$

So on average, the barber will serve $\lambda(1 - q_{10}) = 20(1 - 0.75) = 5$ customers.

2. From the *M/M/1/K* equation, we have:

$$L = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{4}{1-4} - \frac{11 \times 4^{11}}{1-4^{11}} = 9.67$$

So on average, we have 9.67 customers in the shop.

Queuing Practice Problem 1

AN automated warehouse facility of a large retail system services requests for different items from retailers continuously and operates 24 hours daily. The arrival of the requests follows Poisson distribution with 10 requests per hour in average. The warehouse has an automatic crane to retrieve the items from the storage area. The time of service follows exponential distribution and the mean service time is 5 minutes. Each item (including the one being retrieved by the crane), if not delivered immediately, will cost the company $200 per day in average due to penalty or loss of sale. Assume that the system is at its steady state.

a) Determine the utilization rate of the crane.
b) If the company replaces the current crane by a new and faster crane, the new crane can reduce the average service time from 5 to 4 minutes. The additional cost (after factoring the fixed costs) to run the new crane is $500 per day, decide if the company can save money after replacing the current crane by the new one.

***Solution*.**

Arrival 10 req/hr
Mean 5 min.

Mean service time of exponential distribution is $\frac{1}{\mu}$ = 5 min

$$\mu = 12hr$$

a) Utilization $\rho = \frac{\lambda}{\mu} = \frac{10}{12} = 0.8333$
b) Mean service time of exponential distribution is $\frac{1}{\mu}$ = 4 min
c) $\mu = 15hr$

L(current) = $L = \frac{\rho}{1-\rho} = \frac{0.8333}{1-0.8333} = 5$

L(new) = $L = \frac{\rho}{1-\rho} = \frac{10/15}{1-(10/15)} = 2$

(5-2)$200 = $600 Saving, therefore, the company saves $600-$500 = $100 per day with new crane.

Practice Problem

The following probability transition rate matrix of a 3 state continuous parameter stationary Markov process is not complete:

a) Make the matrix complete by filling in the missing diagonal values
b) Write the Kolmogorov Forward System of Differential Equations for this process. (DO NOT SOLVE).

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \mathbf{P} = & \left[\begin{array}{ccc} -- & 0.5 & 0.00 \\ 0.8 & -- & 1.2 \\ 0.0 & 0.6 & -- \end{array}\right] \left.\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right\} \end{array}$$

Solution

$$\mathbf{P} = \left[\begin{array}{ccc} -0.5 & 0.5 & 0.00 \\ 0.8 & -2.0 & 1.2 \\ 0.0 & 0.6 & -0.6 \end{array}\right] \left.\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right\}$$

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}(t)\mathbf{\Lambda}$$

$$\mathbf{P}'(t) = \begin{bmatrix} p'_{00}(t) & p'_{01}(t) & p'_{02}(t) \\ p'_{10}(t) & p'_{11}(t) & p'_{12}(t) \\ p'_{20}(t) & p'_{21}(t) & p'_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} -0.5 & 0.5 & 0.00 \\ 0.8 & -2.0 & 1.2 \\ 0.0 & 0.6 & -0.6 \end{bmatrix}$$

$$p'_{00}(t) \; = -0.5 p'_{00}(t) + 0.8 p'_{01}(t)$$
$$p'_{01}(t) = 0.5 p'_{00}(t) - 2\, p'_{01}(t) + 0.6 p'_{02}(t)$$
$$p'_{02}(t) = 1.2 p'_{01}(t) - 0.6\;\; p'_{02}(t)$$

Practice Problem

The transition probability matrix of a discrete stationary Markov process with 2 states is given:

$$\begin{array}{ccc} & 1 & 2 \\ \left\{\begin{matrix}1\\2\end{matrix}\right. & \multicolumn{2}{c}{\begin{bmatrix}0.2 & 0.8\\0.7 & 0.3\end{bmatrix}} \end{array}$$

a) probability that the process is at state 2 and will be at state 2 again after 2 time periods.

b) probability that the process is at state 1 and will return to state 1 after 3 time periods.

c) probability that the process will be at state 2 in the long run.

Solution

a) $P^2_{22} = (0.7*0.8)+(0.3*0.3) = 0.65$
b) $P^3_{11} = P_{12} * P^2_{21} = (0.8*0.7*0.3) = 0.168$
c) From $\mathbf{v} = \mathbf{vP}$, we have: $[v_1, v_2] = [v_1, v_2]\begin{bmatrix}0.2 & 0.8\\0.7 & 0.3\end{bmatrix}$

Then:

$$v_1 = 0.2v_1 + 0.7v_2$$
$$v_2 = 0.8v_1 + 0.3v_2$$

Replacing the second equation with $v_1 + v_2 = 1$ we have

$$v_1 = 0.2v_1 + 0.7v_2$$
$$v_1 + v_2 = 1$$

Solving: $v_1 = 0.467$
$v_2 = 0.533$

We have $v_1 = 2/3, v_2 = 1/3$**.**

Sample Formula Sheet

1. Summation of number series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, 0 < 1 < x; \qquad \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

2. Formulas related to discrete parameter Markov processes

2.1. $\mathbf{v} = \mathbf{vP}$

2.2. $f_{ij}^{(k)} = \sum_{l \neq j} p_{il} f_{lj}^{(k-1)}, k = 2,3,4,\ldots$ or $\mathbf{f}^{(k)} = \mathbf{Qf}^{(k-1)} = \mathbf{Q}^{k-1}\mathbf{f}^{(1)}$

where $\mathbf{Q}$ is the transition matrix with the corresponding column replaced by vector $\mathbf{0}$.

3. Formulas related to continuous parameter Markov processes

3.1. $\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}(t)\mathbf{\Lambda}, \mathbf{P}(0) = \mathbf{I}$ or $\frac{d\mathbf{P}(t)}{dt} = \mathbf{\Lambda P}(t), \mathbf{P}(0) = \mathbf{I}$

3.2. $\mathbf{q\Lambda} = \mathbf{0}$

4. Queuing models

4.1. Little's Law: $W = L/\lambda$, $W_q = L_q/\lambda$

4.2. *M*/*M*/1 formula

$$\rho = \frac{\lambda}{\mu}, q_i = (1-\rho)\rho^i, i = 0,1,2,3,\ldots \; L = \frac{\rho}{1-\rho}, \; L_q = \frac{\rho^2}{1-\rho}$$

4.3. *M*/*M*/*r* ($r > 1$) formula

$$\rho = \frac{\lambda}{r\mu}, q_0 = \frac{1}{\sum_{n=0}^{r-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^r}{r!} \frac{1}{1-\lambda/(r\mu)}},$$

$$q_n = \begin{cases} \frac{\lambda^n}{n!\,\mu^n} q_0, & \text{if } n \le r, \\ \frac{\lambda^n}{r!\,r^{n-r}\mu^n} q_0, & \text{if } n > r. \end{cases} \qquad L = \frac{q_0(\lambda/\mu)^r \rho}{r!(1-\rho)^2} + \frac{\lambda}{\mu}$$

4.4. *M*/*M*/*1*/*K* formula

$$q_0 = \frac{1-(\lambda/\mu)}{1-(\lambda/\mu)^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}$$

$$q_n = \rho^n \frac{1-\rho}{1-\rho^{K+1}} =, n = 1,2,3,\ldots,K$$

$$L = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}$$