MAT 1322 A Winter 2017 February 8th, 8 :30

Professor Paul-Eugène Parent

A

A 7(b)

4(b)

TEST #1

Name : [redacted]

Student number : [redacted]

DGD registered in (circle yours) : 8 :30 - 10 :00 - 11 :30 - 13 :00 - 14 :30

- Length : 80 min.
- It is a closed book exam.
- You have to answer all questions.
- You have to justify all your answers with a clear and complete solution.
- Answer directly on the exam and you can use the back of each page for your preliminary computations. **Scrap paper is forbidden.**
- **The Faculty approved calculators may be used. All others are forbidden.**
- Cellular phones and unauthorized electronic devices are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur : academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature : [redacted]

$-\frac{1}{x} \Rightarrow \frac{1}{x^2}$

$-(x)^{-1}$

1. (a) (2 points) Consider the integral $\int_0^1 \frac{e^{-1/x}}{x^2}\,dx$. Is it an improper integral (**justify**)? Does it converge? If yes, then compute it.

It is improper because the lower bound, zero, is not in the domain of $\frac{e^{-1/x}}{x^2}$, as it is undefined there.

$$\int_0^1 \frac{e^{-1/x}}{x^2}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\cdot e^{-\frac{1}{x}}dx$$

let $u = -\frac{1}{x}$

$du = \frac{1}{x^2}dx$

$dx = x^2 du$

$\frac{-1}{1} \cdot \frac{-t}{1}$

$$= \lim_{t\to 0^+}\int_{-\frac{1}{t}}^{-1} e^u \cancel{x^2}\,\frac{du}{\cancel{x^2}}$$

1

$$= \lim_{t\to 0^+}\left[\ln\left|-\frac{1}{x}\right|\right]_{-\frac{1}{t}}^{-1}$$

$$= \lim_{t\to 0^+}\left(\cancel{\ln(1)}^{\,0} - \ln(t)\right)$$

$= \infty$ ∴ The improper integral diverges.

(b) (1 point) With the comparison principle determine if the following improper integral

$$\int_1^\infty \frac{\arctan(x)}{x+e^x}\,dx$$

converges or not.

I will find the convergency of $\frac{\pi}{2}\cdot\frac{1}{x+e^x}$ and then use the C. P. to see if $\frac{\arctan(x)}{x+e^x}$ converge or not.

$g(x) = \frac{\pi}{2}\cdot\frac{1}{x+e^x}$

$-\frac{\pi}{2} \le \arctan(x) \le \frac{\pi}{2}$

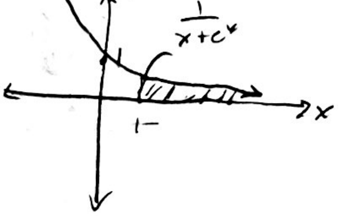
$$\therefore \frac{-\frac{\pi}{2}}{x+e^x} \le \frac{\arctan(x)}{x+e^x} \le \boxed{\frac{\frac{\pi}{2}}{x+e^x}}$$

$$\int_1^\infty \frac{\pi}{2}\frac{1}{x+e^x}dx = \frac{\pi}{2}\cdot\lim_{t\to\infty}\int_1^t \frac{1}{x+e^x}dx$$

$0 \le \frac{1}{x+e^x} \le \infty$

Since $\frac{1}{x+e^x}$ is convergent on $\int_1^\infty$ ∴ $\frac{\pi}{2}\cdot\frac{1}{x+e^x}$ is also convergent on $\int_1^\infty$. And since $\frac{\pi}{2}\cdot\frac{1}{x+e^x} \ge \frac{\arctan(x)}{x+e^x}$, ∴ by the comparison principle $\frac{\arctan(x)}{x+e^x}$ must also be convergent.

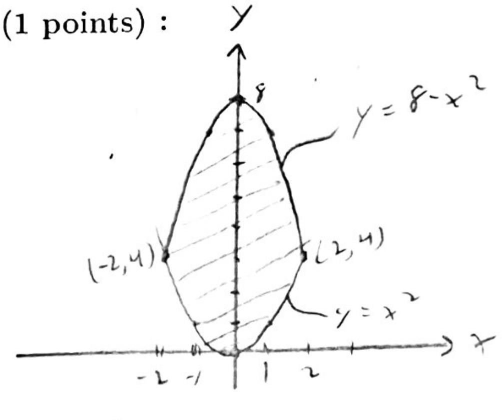
sketch:



0,5

2. (3 points) Draw out the region bounded by the two curves $y = x^2$ et $y = 8 - x^2$. Compute the coordinates of the centroid (center of mass) of that region. Divide your work in four parts :

(a) Sketch of the region (1 points) :



(b) Area of the region (1 points) :

$$A = \int_{-2}^{2} 8 - x^2 - x^2 \, dx$$

$$= \int_{-2}^{2} -2x^2 + 8 \, dx$$

$$= -2\int_{-2}^{2} x^2 - 4 \, dx$$

$$= -2\left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$

$$= -2\left\{\left(\frac{2^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)\right\}$$

$$= -2\left(\left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)\right)$$

$$= 21.33...$$

$$\therefore A \approx 21.3$$

(c) x-coordinate of the centroid (0.5 points) :

This shape is symmetrical through both the x and y axis. ∴ The x-coordinate of the centroid is 0

However if you used $\bar{x} = \dfrac{\int_{-2}^{2} x(f(x) - g(x))\,dx}{A}$ you would come to the same conclusion, that the x-coordinate would be zero.

(d) y-coordinate of the centroid (0.5 points) :

This shape is symmetrical through both the x & y axis. ∴ The y-coordinate of the centroid is $\frac{(8-0)}{2} = 4$

∴ The location of the centroid is (0,4)

If you used $\bar{y} =$

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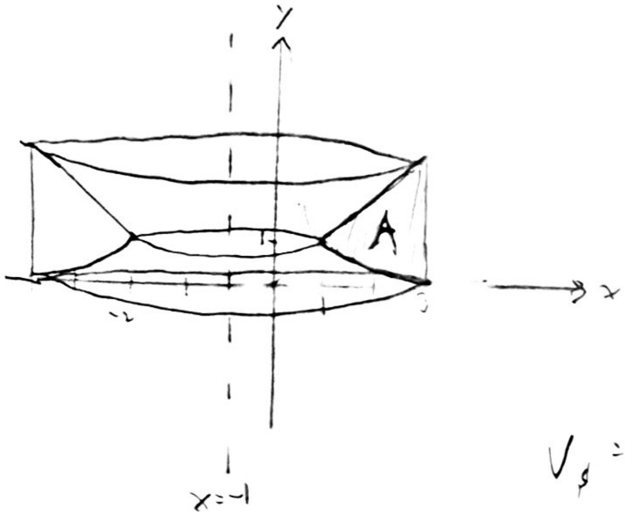
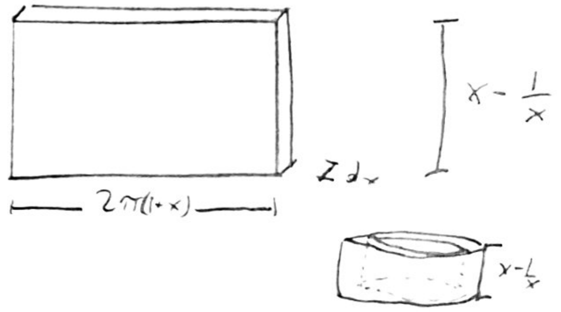
3. (3 points) With the cylindrical shell method compute the volume of the solid of revolution obtained by rotating about the $x = -1$ axis the surface A in the xy-plane bounded by the curves $y = x$, $y = 1/x$, $x = 1$ et $x = 3$. Draw the surface A and a typical elementary cylinder.





$$V_s = \int_1^3 2\pi(1+x)\left(x - \frac{1}{x}\right) dx$$

$$V_s = 2\pi \int_1^3 x - \frac{1}{x} + x^2 - 1 \, dx$$

$$= 2\pi \int_1^3 x^2 + x - \frac{1}{x} - 1 \, dx$$

$$= 2\pi\left(\int_1^3 x^2 + x - 1 \, dx - \int_1^3 \frac{1}{x} dx\right)$$

$$= 2\pi\left(\frac{x^3}{3} + \frac{x^2}{2} - x \Big|_1^3 - \ln x \Big|_1^3\right)$$

$$= 2\pi\left(\left(\left(\frac{3^3}{3} + \frac{3^2}{2} - 3\right) - \left(\frac{1}{3} + \frac{1}{2} - 1\right)\right) - (\ln 3 - \ln 1)\right)$$

$$= 2\pi\left((9 + 4{,}5 - 3) - (-0{,}1\bar{6}) - \ln 3\right)$$

$$= 2\pi\left(\frac{32}{3} - \ln 3\right)$$

$\therefore$ Volume is $2\pi\left(\frac{32}{3} - \ln 3\right)$,
(which is positive).

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