Assignment 3

Due date: Friday, 19 October 2010, 3pm

Total number of points: 13

Q1. Suppose that the random variable X has the following cumulative distribution function CDF:

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x^3, & 0 \le x \le 1 \\ 1, & x \ge 1. \end{cases}$$

(a) Compute $P(X > 0.5)$ and $P(0.2 < X < 0.8)$.

(b) Find the probability density function of X.

(c) Find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.

Solution to Q1:

(a) $P(X > 0.5) = 1 - F(0.5) = 1 - 0.5^3 = 0.875$,
$P(0.2 < X < 0.8) = F(0.8) - F(0.2) = (0.8)^3 - (0.2)^3 = 0.504$.

(b)

$$f_X(x) = F'(x) = 3\,x^2,\ 0 < x < 1.$$

(c)

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 3\,x^3\, dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$$

and

$$\mathrm{E}[X^2] = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \int_0^1 3\,x^4\, dx = \frac{3}{5}x^5\Big|_0^1 = \frac{3}{5}.$$

Thus, $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2 = (3/5) - (3/4)^2 = 3/80 = 0.0375$.

Marking scheme for Q1:

1 point for each correct answer in part (a), 1 point for part (b), 1 point for each correct answer in part (c). Total - 5 points.

Q2. Assume that arrivals of small aircrafts at an airport can be modeled by a Poisson random variable with average 1 aircraft per hour.

(a) What is the probability that more than 3 aircrafts arrive within an hour?

(b) Consider 15 consecutive and disjoint one hour intervals. What is the probability that in none of these intervals we have more than 3 aircraft arrivals?

(c) What is the probability that exactly three aircrafts arrive within 2 hours?

Solution to Q2:

(a) Let X be the number of aircrafts within one hour. Thus, $X \sim Poi(\lambda)$ with $\lambda = \rho\, t = 1\,(1) = 1$. So,

$$P(X > 3) = 1 - [e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} + e^{-1}\frac{1^2}{2!} + e^{-1}\frac{1^3}{3!}] = 0.01899.$$

(b) Let Y be the number of one hour intervals with more than 3 arrivals. Thus, $Y \sim B(15, 0.9197)$. Therefore,

$$P(Y = 0) = \binom{15}{0}(0.01899)^0(1 - 0.01899)^{15} \approx 0.7501.$$

(c) Let W be the number of arrivals within 2 hours. Thus, $W \sim Poi$ with $\lambda = \rho\, t = 1\,(2) = 2$. Thus,

$$P(W = 3) = e^{-2}\,\frac{2^3}{3!}.$$

Marking scheme for Q2:

1 point for each correct answer. Total - 3 points.

Q3. Assume that X is normal with mean 10 and standard deviation 3. Find the value x such that

(a) $P(X > x) = 0.5$

(b) $P(X > x) = 0.95$

(c) $P(x < X < 10) = 0.2$

(d) $P(-x < X - 10 < x) = 0.95$

(e) $P(-x < X - 10 < x) = 0.99$

Solution to Q3:

Using Table III:

(a) $0.5 = P(X > x) = 1 - \Phi[(x-10)/3] \;\Rightarrow\; \Phi[(x-10)/3] = 0.5.$
Thus, $(x-10)/3 = 0 \;\Rightarrow\; x = 10.$

(b) $0.95 = P(X > x) = 1 - \Phi[(x-10)/3] \;\Rightarrow\; \Phi[(x-10)/3] = 0.05$
Thus, $(x-10)/3 = -1.64 \;\Rightarrow\; x = 5.08.$

(c) $0.2 = P(x < X < 10) = \Phi[(10-10)/3] - \Phi[(x-10)/3]$
Thus, $\Phi[(x-10)/3] = \Phi(0) - 0.2 = 0.3.$
Thus, $(x-10)/3 = -0.52 \;\Rightarrow\; x = 8.44.$

(d) $0.95 = P(-x < X - 10 < x) = \Phi(x/3) - \Phi(-x/3) = \Phi(x/3) - [1 - \Phi(x/3)]$
Thus, $\Phi(x/3) = (0.95+1)/2 = 0.975.$
Thus, $x/3 = 1.96 \;\Rightarrow\; x = 5.88.$

(e) $0.99 = P(-x < X - 10 < x) = \Phi(x/3) - \Phi(-x/3) = \Phi(x/3) - [1 - \Phi(x/3)]$
Thus, $\Phi(x/3) = (0.99+1)/2 = 0.995.$
Thus, $x/3 = 2.58 \;\Rightarrow\; x = 7.74.$

Marking scheme for Q3:

1 point for each correct answer. Total - 5 points.