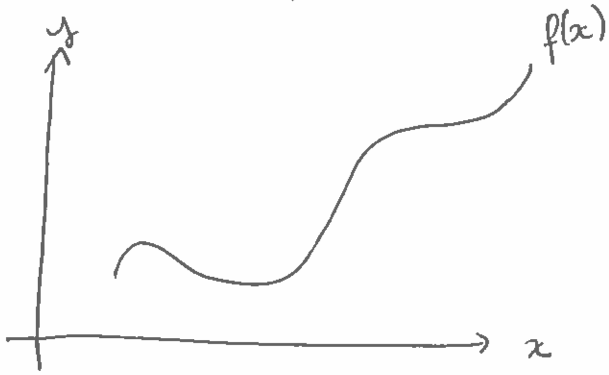
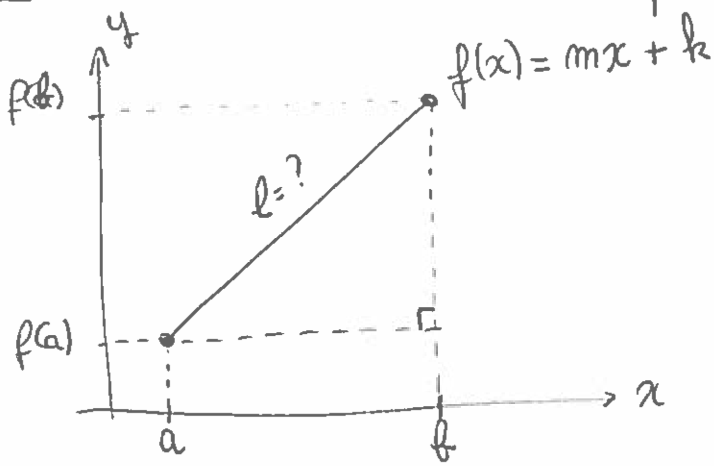
(6) Arc length (§8.1)

What is the length of a curve?



Ex 1: we know how to compute the length of a segment of line



By the Pythagorean theorem

$$\ell = \sqrt{(b-a)^2 + (f(b)-f(a))^2}$$

let us simplify this expression using $f(x) = mx+k$

$$\ell = \sqrt{(b-a)^2 + (mb+k-(ma+k))^2}$$

$$= \sqrt{(b-a)^2(1+m^2)}$$

$$= \underbrace{(b-a)}_{\text{length of the interval}} \times \sqrt{1+\underbrace{m^2}_{f'(x)}}$$

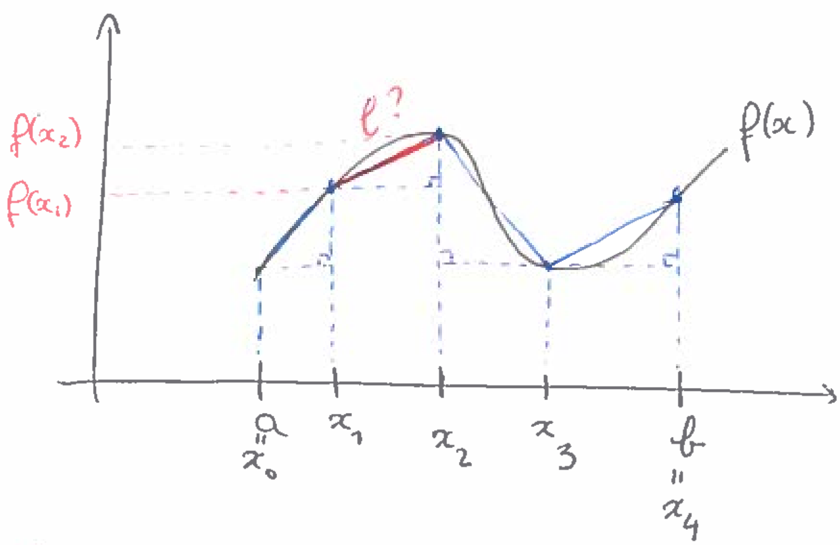
$$= \int_a^b \sqrt{1+f'(x)}\,dx$$

General formula:

If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$ $a \leq x \leq b$ is

$$\boxed{L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx}$$

idea of the proof: approximate the curve by small segments of lines.



$$l = \sqrt{(x_2-x_1)^2 + (f(x_2)-f(x_1))^2}$$

$$\text{total length} \simeq \sum_{i=1}^{4} \sqrt{(x_i-x_{i-1})^2 + (f(x_i)-f(x_{i-1}))^2}$$

we take the limit as the length of the intervals $[x_{i-1},x_i]$ goes to 0 (the number of interval goes to infinity)

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) \quad \text{where } c_i \in [x_{i-1},x_i]$$

Mean value theorem
(uses the fact that f' is continuous).

$$\mathcal{L} = \lim_{m\to\infty} \sum_{i=1}^{m} \sqrt{(x_i - x_{i-1})^2 + \left(f'(c_i)(x_i - x_{i-1})\right)^2}$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} \underbrace{\sqrt{1 + f'(c_i)^2}}_{\simeq\, f'(x_i)} \times \underbrace{(x_i - x_{i-1})}_{\simeq\, dx}$$

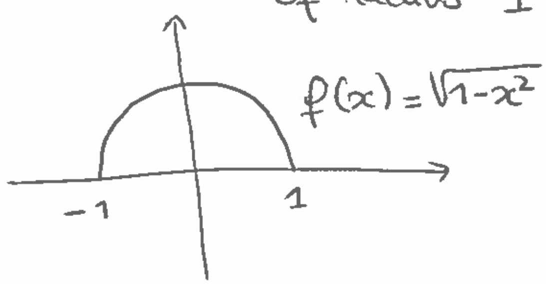
as the x_i, x_{i-1} get closer and closer

$f'(c_i) \approx f(x_i)$

$$\boxed{\mathcal{L} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.}$$

□

Ex 2: [half circle] of radius 1

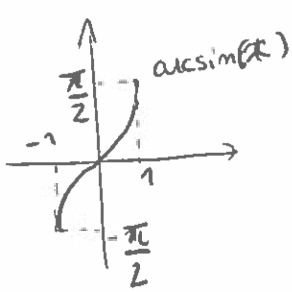


$$\mathcal{L} = \int_{-1}^{1} \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$\mathcal{L} = \int_{-1}^{1} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, dx = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$$

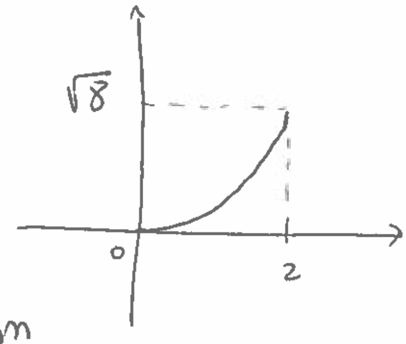
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$$



$$= \left[\arcsin(x)\right]_{-1}^{1}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

<u>Ex 3</u>: $f(x) = x^{3/2}$ arc length from 0 to 2?



$$\mathcal{L} = \int_0^2 \sqrt{1+(f'(x))^2}\,dx$$

$$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{1/2}$$

substitution

$$\begin{cases} u = 1 + \frac{9}{4}x \\ du = \frac{9}{4}dx \\ x = 0 \rightarrow u = 1 \\ x = 2 \rightarrow u = 1 + \frac{9}{2} = \frac{11}{2} \end{cases}$$

$$\mathcal{L} = \int_0^2 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \int_1^{11/2} \frac{4}{9}\sqrt{u}\,du$$

$$= \frac{4}{9}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^{11/2} = \frac{4}{9}\left(\frac{2\left(\frac{11}{2}\right)^{3/2}}{3} - \frac{2}{3}\right)$$

$$\simeq 3.526$$

<u>Rk</u>: the length of the segment of line between $(0,0)$ and $(2, \sqrt{8})$ is

$$L = \sqrt{2^2 + \sqrt{8}^2} = \sqrt{4+8} = \sqrt{12} \simeq 3.464$$

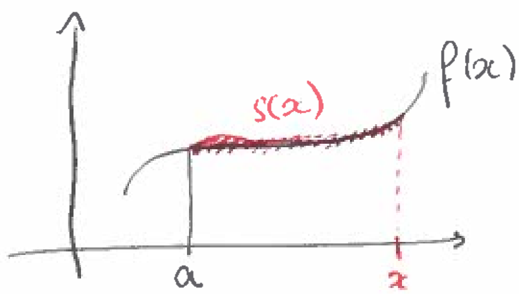
so our result is reasonnable.

* Arc length function

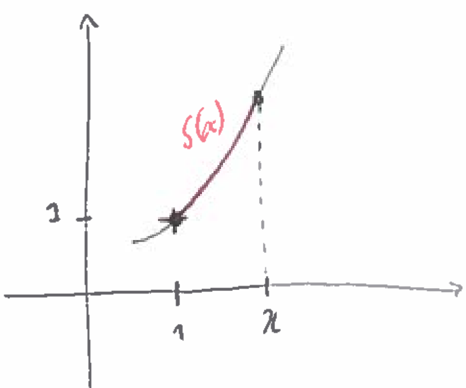
We want to measure the distance on a path as we follow this path.

$$\boxed{s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt}$$

arc length function of a function f from the point "a" to x.
it is a function of x
(the variable of integration is t)



Ex 4: Arc length function for the curve $y = x^2 - \frac{1}{8}\ln(x)$ starting at $(1,1)$



$f(t) = t^2 - \frac{1}{8}\ln(t)$

$f'(t) = 2t - \frac{1}{8t}$

$(f'(t))^2 = 4t - \frac{1}{2} + \frac{1}{64t}$

$$s(x) = \int_1^x \sqrt{1+(f'(t))^2}\,dt$$

$$= \int_1^x \sqrt{4t - \frac{1}{2} + \frac{1}{64t} + 1}\,dt$$

$$= \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\,dt = \int_1^x \left(2t + \frac{1}{8t}\right)dt = x^2 + \frac{1}{8}\ln(x) - 1$$