SOLUTION TO FINAL EXAM

MAT 1322D, Fall 2012
Total = 53 marks

1. (4 marks) Determine whether each of the following improper integrals is convergent or divergent. If it is convergent, find its value; if it is divergent, use appropriate method to justify your conclusion:

(a) $\int_0^\infty \frac{1}{(x+1)(x+3)}\,dx$.

Solution. Use partial fraction, $\frac{1}{(x+1)(x+3)} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x+3}\right)$.

$$\int_0^\infty \frac{1}{(x+1)(x+3)}\,dx = \frac{1}{2}\lim_{b\to\infty}\left(\int_0^b \frac{dx}{x+1} - \int_0^b \frac{dx}{x+3}\right) = \frac{1}{2}\lim_{b\to\infty}\left(\ln(b+1) - \ln(b+3) + \ln 3\right)$$
$$= \frac{1}{2}\lim_{b\to\infty}\left(\ln\frac{b+1}{b+3} + \ln 3\right) = \frac{1}{2}\ln 3.$$

(b) $\int_0^1 \frac{1}{\sqrt{x^2+2x^3}}\,dx$.

Solution. Since $x^3 \le x^2$ in the interval [0, 1], $\frac{1}{\sqrt{x^2+2x^3}} > \frac{1}{\sqrt{3x^2}} = \frac{1}{\sqrt{3}x}$. The improper integral $\int_0^1 \frac{1}{\sqrt{3}x}\,dx = \frac{1}{\sqrt{3}}\int_0^1 \frac{1}{x}\,dx$ is divergent, this integral is divergent.

2. (4 marks) Find the volume of a solid whose base is the region on the x-y plane bounded by the x-axis and the graph of the function $y = 2(1 - x^2)$. The cross sections of the solid perpendicular to the x-axis are semicircles with diameter on the x-y plane. Find the volume of this solid.

Solution. The area of the cross section is $A(x) = \frac{1}{2}\pi(1 - x^2)^2$. The volume of the solid is

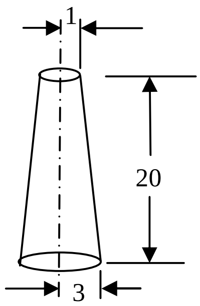
$$V = \frac{1}{2}\pi\int_{-1}^1 (1-x^2)^2\,dx = \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{x=0}^1 = \frac{8}{15}\pi.$$

3. (4 marks) Find the length of the arc $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$, $1 \le x \le 2$.

Solution. $y' = x - \dfrac{1}{4x}$, $(y')^2 = x^2 - \dfrac{1}{2} + \dfrac{1}{16x^2}$. $1 + (y')^2 = x^2 + \dfrac{1}{2} + \dfrac{1}{16x^2} = \left(x + \dfrac{1}{4x}\right)^2$.

The length of the arc is $L = \displaystyle\int_1^2 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{1}{4}\ln x\right]_{x=1}^{2} = \frac{3}{2} + \frac{\ln 2}{4}$.

4. (5 marks) A monument of the shape of a truncated circular cone is built with stone of density 5000 kg / m^3. The radius of the bottom of the monument is 3 meters, the radius of the top is 1 meter, and the height of the monument is 20 meters. (Recall that the radius of the cross section at height h is $3 - h / 10$ meters). Find the work (in Joules) needed to lift the stone from the ground level to build the monument. (Use g = 9.8 m/sec^2)



Solution. A layer of the monument at height h of thickness dh weights

$dw = \delta g \pi (3 - h / 10)^2 dh$.

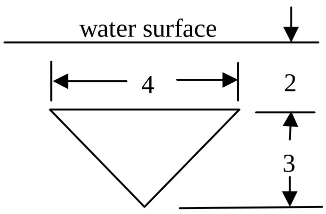
The work needed to lift the stone of this layer from the ground to height h is

$dW = \delta g \pi (3 - h / 10)^2 h dh$.

The total work needed is

$$W = \delta g \pi \int_0^{20} \left(3 - \frac{h}{10}\right)^2 h dh = \delta g \pi \left[\frac{9}{2}h^2 - \frac{1}{5}h^3 + \frac{1}{400}h^4\right]_{h=0}^{20} \approx 2.94 \times 10^7 \text{ Joule.}$$

5. (4 marks) Suppose that a triangular surface is submerged into water as shown in the following figure. The top of the triangle is 2 meters under the water surface. The height of the triangle is 3 meters, and the length of the top side is 4 meters. Find the force acting on this surface.



Solution. Take a slice of the surface at depth 2 + D of thickness dD. The length of this slice is approximately $x = \dfrac{4(3-D)}{3}$. The force acting on this slice is

$$dF = 1000g\,\frac{4(3-D)(2+D)}{3}\,dD\,.$$

The total force is

$$F = \frac{4000g}{3}\int_0^3 (3-D)(2+D)dD = 18000g \approx 176400 \text{ Newton.}$$

6. (5 marks) Consider the initial-value problem: $y' = 2t\cos^2 y$, $y(1) = \pi/4$.

(i) (3 marks) Solve this initial-value problem.

(ii) (2 marks) Use Euler's method with step size h = 0.25 to find an approximation of $y(1.5)$. (Use 4 digits after the decimal point in your calculation).

Solution. (i) Separating variables, $\int \dfrac{dy}{\cos^2 y} = \int 2t\,dt$. tan $y = t^2 + C$. By the initial condition C = 0. Then $y = \arctan(t^2)$.

(ii) $t_0 = 1$, $y(1) = y_0 = \pi/4 \approx 0.7854$.
$t_1 = 0.25$, $y(1.25) \approx y_1 = y_0 + h\,(2t_0 \cos^2 y_0) \approx 1.0354$.
$t_2 = 0.50$, $y(1.50) \approx y_2 = y_1 + h\,(2t_1 \cos^2 y_1) \approx 1.1981$.

7. (6 marks) Consider the equation $\dfrac{dy}{dt} = y\,(5 - y)$, $y(0) = 2$.

(i) (3 marks) Solve this equation analytically.

(ii) (1 mark) Find the limit of the solution when t approaches infinity.

(iii) (2 marks) Sketch the graph of the solution of this initial-value problem. Mark the inflection point of the graph.

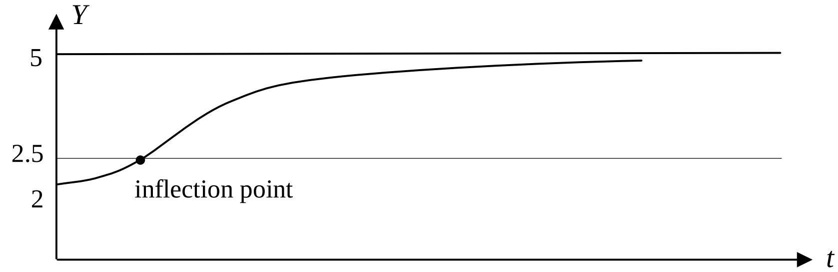
Solution. (i) Separating variables, $\int \dfrac{dy}{y(5-y)} = \int dt$. Use partial fraction,

$$\int \frac{dy}{y(5-y)} = \frac{1}{5}\left(\int \frac{1}{y}dy + \int \frac{1}{5-y}dy\right) = \frac{1}{5}\ln\left|\frac{y}{5-y}\right| = t + C .$$

$\frac{y}{5-y} = Ke^{5t}$. By the initial condition, $K = \frac{2}{3}$. Hence, $y = \frac{10e^{5t}}{3+2e^{5t}} = \frac{10}{2+3e^{-5t}}$.

(ii) The limit of the solution is 10 / 2 = 5.

(iii) The graph of the solution looks like the following



8. (6 marks) Determine whether each of the following series is convergent or divergent. Justify your answer by appropriate test method and state the condition to use these tests:

(a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$; (b) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln n}$; (c) $\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{n^3+n}}$.

Solution. (a) Since the general term is positive, decreasing and continuous, we can the integral test.

$\int_0^{\infty} xe^{-x}dx = \lim_{b\to\infty}\left[-(1+x)e^{-x}\right]_{x=0}^{b} = \lim_{b\to\infty}(1-(b+1)e^{-b}) = 1 < \infty$, this series is convergent.

Since the series is positive, we may also use the ratio test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)e^n}{ne^{n+1}} = \lim_{n\to\infty}\frac{1}{e} < 1.$ By the ratio test, this series converges.

(b) Since the general term is alternating and decreasing, we can use the alternating series test. Since the general term approaches 0, this series is convergent.

(c) Since this is a positive series, we can use the limit comparison test.

Compare this series with the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. We have

$$\lim_{n\to\infty}\frac{2n-1}{\sqrt{n^3+n}}\sqrt{n}=\lim_{n\to\infty}\frac{\sqrt{n}(2n-1)/(n\sqrt{n})}{\sqrt{n^3+n}/(n\sqrt{n})}=\lim_{n\to\infty}\frac{2-1/n}{\sqrt{1+1/n^2}}=2\,.$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with p = 1 / 2, it is divergent. Series $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n}}$ is also divergent.

9. (4 marks) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(x+5)^n}{3^n n}\,.$$

Solution. The center of the series is –5. The radius of convergence is

$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)}{3^n n}\right|=3$. This series is absolutely convergent in the interval (–8, –2).

When x = –8, the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{3^n n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. This is an alternating decreasing series with general term approaching 0. By the alternating series test, this series is convergent.

When x = –2, the series is $\sum_{n=1}^{\infty}\frac{3^n}{3^n n}=\sum_{n=1}^{\infty}\frac{1}{n}$. It is the harmonic series, which is divergent. The interval of convergence of this series is [–8, –2).

10. (5 marks) Recall that the binomial series is

$$(1+x)^k=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)...(k-n+1)}{n!}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+...$$

(i) (2 marks) Find the Maclaurin series of the function $y=\frac{1}{\sqrt{1+x^2}}$.

(ii) (3 marks) Use the fact that $\frac{d}{dx}\ln(x+\sqrt{1+x^2})=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}$ to find the Maclaurin series of the function $y=\ln(x+\sqrt{1+x^2})$.

You may use or not use the sigma notation. If you don't use the sigma notation, give enough number of terms to demonstrate the answer.

Solution. $\dfrac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = \displaystyle\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} x^{2n}$

$=1+\left(-\frac{1}{2}\right)x^2+\frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^4+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^6+\ldots$

$=1-\frac{1}{2}x^2+\frac{1\cdot 3}{2!2^2}x^4-\frac{1\cdot 3\cdot 5}{3!2^3}x^6+\ldots$

$= \displaystyle\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{n!2^n}x^{2n}$.

$\ln(x+\sqrt{1+x^2}) = \displaystyle\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{n!2^n(2n+1)}x^{2n+1}$

$= x-\frac{1}{2\cdot 3}x^3+\frac{1\cdot 3}{2!2^2\cdot 5}x^5-\frac{1\cdot 3\cdot 5}{3!2^3\cdot 7}x^7+\ldots = x-\frac{1}{6}x^3+\frac{3}{40}x^5-\frac{5}{112}x^7+\ldots$

11. (6 marks) Consider the 2-variable function $z = f(x, y) = \dfrac{xy}{2x^2+y^2+1}$.

(a) (2 mark) Find the gradient vector of this function at the point (2, 1).

(b) (2 marks) Find the directional derivative of z at point (2, 1) in the direction of the vector $\mathbf{v} = 3\mathbf{i}+4\mathbf{j}$.

(c) (2 marks) Find the equation of the plane tangent to the graph of this function at a point where $x = 2$, and $y = 1$.

Solution. (a) $z_x = \dfrac{y(2x^2+y^2+1)-4x^2y}{(2x^2+y^2+1)^2} = \dfrac{y^3-2x^2y+y}{(2x^2+y^2+1)^2}$.

$z_y = \dfrac{x(2x^2+y^2+1)-2xy^2}{(2x^2+y^2+1)^2} = \dfrac{2x^3-xy^2+x}{(2x^2+y^2+1)^2}$.

When $x = 2$, $y = 1$, $z_x = \dfrac{-6}{100} = -\dfrac{3}{50}$, and $z_y = \dfrac{16}{100} = \dfrac{4}{25}$.

(c) The unit vector in the direction $\mathbf{v}$ is $\mathbf{u} = \frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$. The directional derivative is

$$D_{\mathbf{v}}(z) = -\frac{6}{100}\times\frac{3}{5}+\frac{16}{100}\times\frac{4}{5} = \frac{46}{500} = \frac{23}{250}.$$

(d) When $x = 2$, and $y = 1$, $z = \frac{1}{5}$. The equation of the tangent plane is

$z = -\frac{3}{50}(x-2)+\frac{4}{25}(y-1)+\frac{1}{5}$, or $50z = -3x + 8y + 8$, $3x - 8y + 50z = 8$.