MATH 205 Final Examination
December 2010

SOLUTIONS

MARKS

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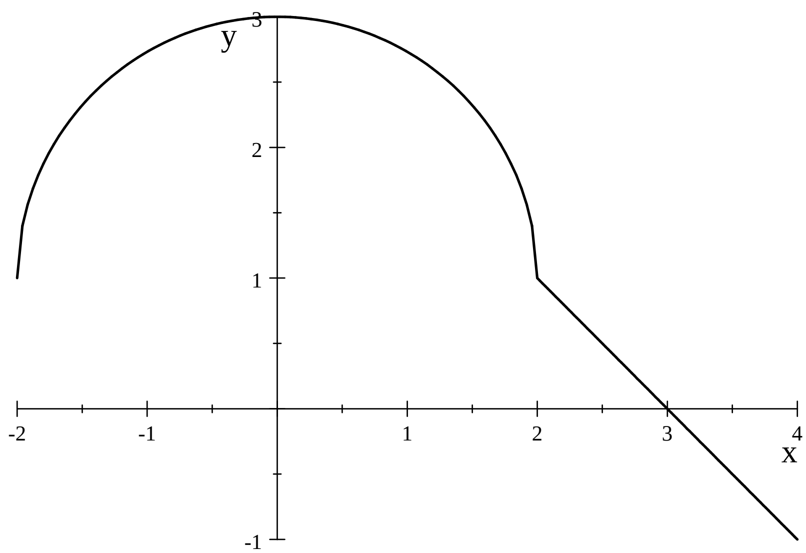
1 (a)

$$f(x) = \begin{array}{cc} 1+\sqrt{4-x^2} & \text{if} \quad |x| \le 2 \\ 3-x & \text{if} \quad |x| > 2 \end{array}$$

Here is the graph:



To evaluate $I = \int_{-2}^{4} f(x)$ in terms of areas we break it up into $I = \int_{-2}^{2} f(x) + \int_{2}^{4} f(x)$. Clearly $\int_{-2}^{2} f(x) = A_1$ is the area of the half disk of radius 2 plus the area of the 4×1 rectangle on which it sits $= \dfrac{1}{2}\pi \cdot 2^2 + 4 = 2\pi + 4$. The second integral is 0 because there are two congruent 1×1 triangles: one above and one below the x-axis. So $I = 2\pi + 4$.

1 (b) If $F(x) = \int_0^{\sqrt{x}} \sqrt{1+t^2} \cos\left(\pi t^2\right) dt$, the derivative is calculated using the Chain Rule, where we first put $u = \sqrt{x}$ and write $F(x) =$

$\int_0^u \sqrt{1+t^2}\cos\left(\pi t^2\right) dt$:

$$\begin{aligned} F'(x) &= \frac{d}{du}(F(u)) \cdot \frac{du}{dx} \\ &= \sqrt{1+u^2}\cos\left(\pi u^2\right)\frac{1}{2}x^{-1/2} \text{ (by the FTC)} \\ &= \frac{1}{2\sqrt{x}}\sqrt{1+x}\cos(\pi x) \text{ and} \\ F'(1) &= \frac{1}{2}\sqrt{2}\cdot(-1) = -\sqrt{2} \end{aligned}$$

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2 (a) Let $I = \int \frac{e^x}{4+e^{2x}} dx$ and use the substitution $u = e^x$; $du = e^x dx$. Then

$$\begin{aligned} I &= \int \frac{e^x}{4+e^{2x}} = \int \frac{du}{4+u^2} \\ &= \frac{1}{2}\int \frac{d(u/2)}{1+(u/2)^2} = \frac{1}{2}\arctan(u/2) + C \\ &= \frac{1}{2}\arctan(e^x/2) + C \end{aligned}$$

2 (b) Let $I = \int \frac{(1-\sqrt{x})^2}{x^{3/2}} dx$ and expand the numerator first:

$$\begin{aligned} I &= \int \frac{(1-\sqrt{x})^2}{x^{3/2}} dx = \int \frac{1-2x^{1/2}+x}{x^{3/2}} dx \\ &= \int \left(x^{-3/2} - 2x^{-1} + x^{-1/2}\right) dx \\ &= -2x^{-1/2} - 2\ln|x| + 2x^{1/2} + C \end{aligned}$$

2 (c) Let $I = \int \frac{1+x}{x^3-4x} dx = \int \frac{1+x}{x(x-2)(x+2)} dx$ Using partial fractions we have

$$\begin{aligned} \frac{1+x}{x(x-2)(x+2)} &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \\ &= \frac{A\left(x^2-4\right) + Bx(x+2) + Cx(x-2)}{x(x-2)(x+2)} \end{aligned}$$

Substitute $x = 0$ to get $1 = -4A$ so $A = -1/4$; substitute $x = 2$ to get $3 = 8B$ so $B = 3/8$ and finally substitute $x = -2$ to get $-1 = 8C \Rightarrow C = -1/8$. So

$$\begin{aligned} I &= -\frac{1}{4}\int \frac{dx}{x} + \frac{3}{8}\int \frac{dx}{x-2} - \frac{1}{8}\int \frac{dx}{x+2} \\ &= -\frac{1}{4}\ln|x| + \frac{3}{8}\ln|x-2| - \frac{1}{8}\ln|x+2| + C \end{aligned}$$

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3 (a) Let $I = \int \dfrac{\ln x^2}{x^2} dx = 2\int \dfrac{\ln x}{x^2} dx$. We integrate by parts:

$$u = 2\ln x \qquad dv = \frac{dx}{x^2}$$
$$du = \frac{2}{x} \qquad v = -\frac{1}{x}$$

and so $I = -2\dfrac{\ln x}{x} + 2\int \dfrac{dx}{x^2} = -2\dfrac{\ln x}{x} - \dfrac{2}{x} + C$. So

$$\begin{aligned}\int_1^e \frac{\ln x^2}{x^2} dx &= -\left(2\frac{\ln x}{x} + \frac{2}{x}\right)\Big|_1^e = -\left(\frac{2}{e} + \frac{2}{e}\right) + (2) \\ &= 2 - \frac{4}{e}\end{aligned}$$

3 (b) Let $I = \int (1 - \sin x)\cos^3 x dx$ and put $u = \sin x$, $du = \cos x dx$. Then

$$\begin{aligned}I &= \int (1 - \sin x)\cos^2 x \cos x dx \\ &= \int (1 - \sin x)\left(1 - \sin^2 x\right)\cos x dx \\ &= \int (1 - u)\left(1 - u^2\right) du = \int \left(1 - u - u^2 + u^3\right) du \\ &= u - u^2/2 - u^3/3 + u^4/4\end{aligned}$$

and to find $\int_0^{\pi/2} (1 - \sin x)\cos^3 x dx$ we note that when $x = 0$, $u = 0$ and $x = \pi/2$, $u = 1$ so the answer is

$$\left(u - u^2/2 - u^3/3 + u^4/4\right)\Big|_0^1 = 1 - 1/2 - 1/3 + 1/4 = 5/12$$

3 (c) Let $I = \int_0^2 \sqrt{4 - x^2} dx$. This can be done using the trigonometric substitution $x = 2\sin t$, etc. but a **much easier way** is to notice that this integral represents the area of a quarter circle centered at the origin with radius 2, so the answer is

$$\frac{1}{4}\pi \cdot 2^2 = \pi$$

8

4 (a) Let $I = \int_{-\infty}^0 x^3 e^{-2x^2} dx = \lim_{t\to-\infty} \int_t^0 x^3 e^{-2x^2} dx$. Put $u = -2x^2$ and so $du = -4xdx$; $xdx = -1/4du$. Now when $x = 0$, $u = 0$ and as

$x \to -\infty$, $u \to -\infty$ so $I = -\frac{1}{4}\int_{-\infty}^{0} x^2 e^u du$. We can't have x and u mixed so we need to get rid of the remaining x^2. But $x^2 = -\frac{1}{2}u$, so

$$I = \frac{1}{4} \cdot \frac{1}{2} \int_{-\infty}^{0} ue^u du = \frac{1}{8} \lim_{t \to -\infty} \int_{t}^{0} ue^u du$$

Using integration by parts for $\int ue^u du$: Let $U = u$ and $dV = e^u du$; $dU = du$ and $V = e^u$ so $\int ue^u du = ue^u - e^u$ (omit $+C$) so we have

$$I = \frac{1}{8} \lim_{t \to -\infty} \left. (ue^u - e^u) \right|_t^0 = -\frac{1}{8}$$

4 (b) Let $I = \int_0^1 \frac{dx}{(1-x)^{5/4}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{(1-x)^{5/4}}$. Put $u = 1 - x$; $du = -dx$ and $x = 0 \Rightarrow u = 1$; $x = 1 \Rightarrow u = 0$ so

$$I = \int_1^0 \frac{-du}{u^{5/4}} = \int_0^1 \frac{du}{u^{5/4}}$$

Since this is a Type II integral with integrand $\frac{1}{u^p}$ with $p > 1$, it diverges (you don't have to prove it; stating it is enough).

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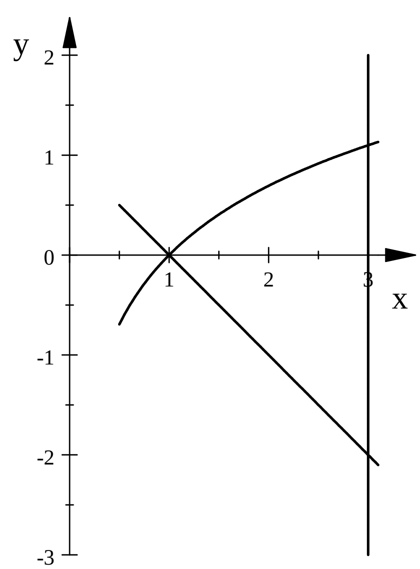
5 (a) The boundaries are $y = \ln x$; $y = 1 - x$; and $x = 3$. The sketch is:



Guided by the sketch we see that $y = \ln x$; and $y = 1 - x$ intersect at $(1, 0)$. Also the upper graph is $y = \ln x$ and the lower one is $y = 1 - x$ so the area is

$$\begin{aligned} A &= \int_1^3 [\ln x - (1 - x)]\, dx = \left[x \ln x - x - \left(x - \frac{x^2}{2} \right) \right] \Bigg|_1^3 \\ &= \left[3 \ln 3 - 3 - \left(3 - \frac{9}{2} \right) \right] - \left[0 - 1 - \left(1 - \frac{1}{2} \right) \right] \\ &= 3 \ln 3 - 6 + \frac{9}{2} + 2 - \frac{1}{2} = 3 \ln 3 \end{aligned}$$

(b) The region bounded by $y = \cos(\pi x)$ and $y = -1$; $0 \leq x \leq 1$ rotated about the horizontal line $y = -$ gives a solid of revolution. For each x the cross-section is a disc with radius $r(x) = 1 + \cos(\pi x)$ and so the cross-sectional area $A(x) = \pi r(x)^2$. So the volume is given by

$$\begin{aligned} V &= \int_0^1 A(x)\, dx = \int_0^1 \pi r(x)^2\, dx \\ &= \pi \int_0^1 (1 + \cos(\pi x))^2\, dx \\ &= \pi \int_0^1 \left(1 + 2\cos(\pi x) + \cos^2(\pi x)\right) dx \\ &= \pi \int_0^1 \left(1 + 2\cos(\pi x) + \frac{1 + \cos(2\pi x)}{2} \right) dx \\ &= \pi \left[x - \frac{2\sin(\pi x)}{\pi} + \frac{1}{2}x - \frac{\sin(2\pi x)}{2 \cdot 2\pi} \right] \Bigg|_0^1 \\ &= \pi \left[\frac{3}{2} - 0 - 0 \right] - \pi [0 - 0 - 0] \\ &= \frac{3\pi}{2} \end{aligned}$$

(c) The mean value of a function $f(x)$ on an interval $[a, b]$ is $m = \frac{1}{b - a} \int_a^b f(x)\, dx$. So here we have

$$\begin{aligned} m &= \frac{1}{\pi/4 - 0} \int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1 + 3\tan x}} dx \\ &= \frac{4}{\pi} \int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1 + 3\tan x}} dx \end{aligned}$$

So let $u = 1 + 3\tan x$; $du = 3\sec^2 x dx$. Then $x = 0 \Rightarrow u = 1$ and

$x = \pi/4 \Rightarrow u = 4$ and we get

$$m = \frac{4}{\pi}\int_1^4 \frac{du}{3\sqrt{u}} = \frac{4}{3\pi}\cdot 2u^{1/2}\Big|_1^4$$
$$= \frac{8}{3\pi}(2-1) = \frac{8}{3\pi}$$

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6 (a) $a_n = \dfrac{(-1)^n n}{\sqrt{1+9n^2}} = \dfrac{(-1)^n}{\sqrt{9+1/n^2}}$ and the denominator $\to 3$ but the numerator has no limit so the limit does not exist.

6 (b) $a_n = \dfrac{\ln\left(1+e^{2n}\right)}{1+n^2}$. Since this is of the form $\dfrac{\infty}{\infty}$ we can change the variable to x and apply L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln\left(1+e^{2x}\right)}{1+x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{\left(1+e^{2x}\right)(2x)}$$
$$= \lim_{x\to\infty}\frac{2e^{2x}}{2x+2xe^{2x}}$$
$$= \lim_{x\to\infty}\frac{2}{2xe^{-2x}+2x} = 0$$

and so $\lim_{n\to\infty} a_n = 0$

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7 (a) $\sum_{n=1}^{\infty}\dfrac{n2^n}{(1+n)!} = \sum_{n=1}^{\infty} a_n$ and we use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\,2^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{n2^n} = \frac{2\,(n+1)}{n\,(n+2)} \to 0 = L$$

and so since $L < 1$ the series converges absolutely.

(b) $\sum_{n=1}^{\infty}\dfrac{(-1)^n\ln(1+n)}{(1+n)} = \sum_{n=1}^{\infty}(-1)^n a_n$ where $a_n \downarrow$ and $a_n \to 0$ so by the alternating series test is converges. Since $a_n > \dfrac{1}{n+1}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n+1}$ diverges, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n\ln(1+n)}{(1+n)}$ only converges conditionally - i.e. $\sum_{n=1}^{\infty}\dfrac{(-1)^n\ln(1+n)}{(1+n)}$ converges but $\sum_{n=1}^{\infty}\left|\dfrac{(-1)^n\ln(1+n)}{(1+n)}\right| = \sum_{n=1}^{\infty}\dfrac{\ln(1+n)}{(1+n)}$ diverges.

(c) $\sum_{n=1}^{\infty}\dfrac{(-1)^n+n^2}{(1+2n)^2}$ diverges because $\lim_{n\to\infty}\dfrac{(-1)^n+n^2}{(1+2n)^2} = \lim_{n\to\infty}\dfrac{(-1)^n/n^n+1}{(1/n+2)^2} = \dfrac{1}{4}$ and therefore is not 0, which is a required condition for convergence.

5

8 Let $f(x) = \sum_{n=0}^{\infty} \frac{(x^2+1)^n}{3^{n+1}}$. We see that this is a geometric series with $a = \frac{(x^2+1)^0}{3^{0+1}} = \frac{1}{3}$ and $r = \frac{(x^2+1)}{3}$. We also know that for a geometric series to converge we need $|r| < 1$ and so

$$\begin{aligned} \frac{(x^2+1)}{3} &< 1 \\ x^2+1 &< 3 \\ x^2 &< 2 \\ |x| &< \sqrt{2} \end{aligned}$$

and so the radius of convergence is $\sqrt{2}$. The sum of the series is

$$\begin{aligned} f(x) &= \frac{a}{1-r} = \frac{1}{3} \cdot \frac{1}{1 - \left(\frac{x^2+1}{3}\right)} \\ &= \frac{1}{2-x^2}; |x| < \sqrt{2} \end{aligned}$$

6

9 (a) For $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n2^n}$ we first use the ratio test (or it could be the root test)

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{(n+1)\,2^{n+1}} \right| \cdot \left| \frac{n2^n}{(x+5)^n} \right| \\ &= \lim_{n\to\infty} \frac{|x+5|}{2} \cdot \frac{n}{n+1} = \frac{|x+5|}{2} \\ L < 1 &\Leftrightarrow |x+5| < 2 \Leftrightarrow -7 < x < -3 \end{aligned}$$

so the radius of convergence is $R = 2$.

9 (b) Test the endpoints: first at $x = -7$. The series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is conditionally convergent (negative of the alternating harmonic series). Next, put $x = -3$:

$$\sum_{n=1}^{\infty} \frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges (harmonic series). So the interval of convergence is $[-7, -3)$.

5

Bonus To express $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ in terms of "familiar" functions, we use the hint:

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n3^n} \\
f'(x) &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n3^n} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{3^n} \\
&= \frac{1}{3}\sum_{n=1}^{\infty} \frac{x^{n-1}}{3^{n-1}} = \frac{1}{3}\sum_{j=0}^{\infty} \frac{x^j}{3^j} \\
&= \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}}; \left|\frac{x}{3}\right| < 1 \\
&= \frac{1}{3-x}; |x| < 3 \quad \text{and so} \\
f(x) &= \int f'(t)\,dt \\
&= \int \frac{dt}{3-t} \\
&= -\ln|3-t| + C
\end{aligned}$$

Since $f(0) = \sum_{n=1}^{\infty} \frac{0^n}{n3^n} = 0$, we get $0 = -\ln 3 + C$ so $C = \ln 3$ and $f(x) = \ln 3 - \ln|3-x|$. Another way to write this is

$$\begin{aligned}
f(x) &= \ln 3 - \ln|3-x| \\
&= \ln \frac{3}{|3-x|} \\
&= \ln \frac{1}{|1-x/3|} \\
&= -\ln\left|1-\frac{x}{3}\right|
\end{aligned}$$