COEN-231: Introduction to Discrete Mathematics

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Lecture 2.

Arash Mohammadi
Office: S-EV 9187
Email: arashmoh@encs.Concordia.ac
Homepage: http://users.encs.concordia.ca/~arashmoh/

Outline of this Class

Objectives:

- Learn about statements (propositions)
- Learn how to use logical connectives to combine statements
- Explore how to draw conclusions using various argument forms
- Become familiar with quantifiers and predicates
- Learn various proof techniques

References:

Check the following materials on the course website

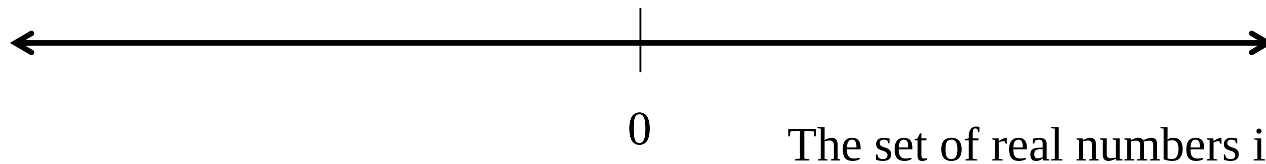
- Hand out General (HO-1)
- Book Sections 1.1 and 1.2

What is Discrete Math?

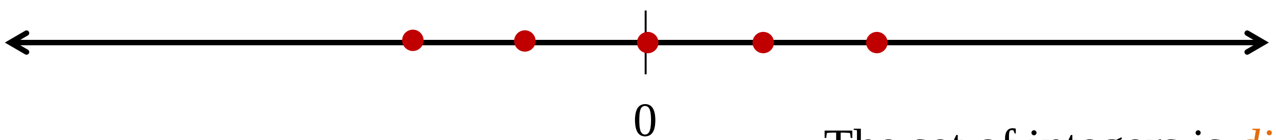
What is Discrete Math?

“Discrete mathematics is the study of **mathematical structures** that are fundamentally **discrete** rather than continuous.”

[Wikipedia.com]



The set of real numbers is *continuous*.

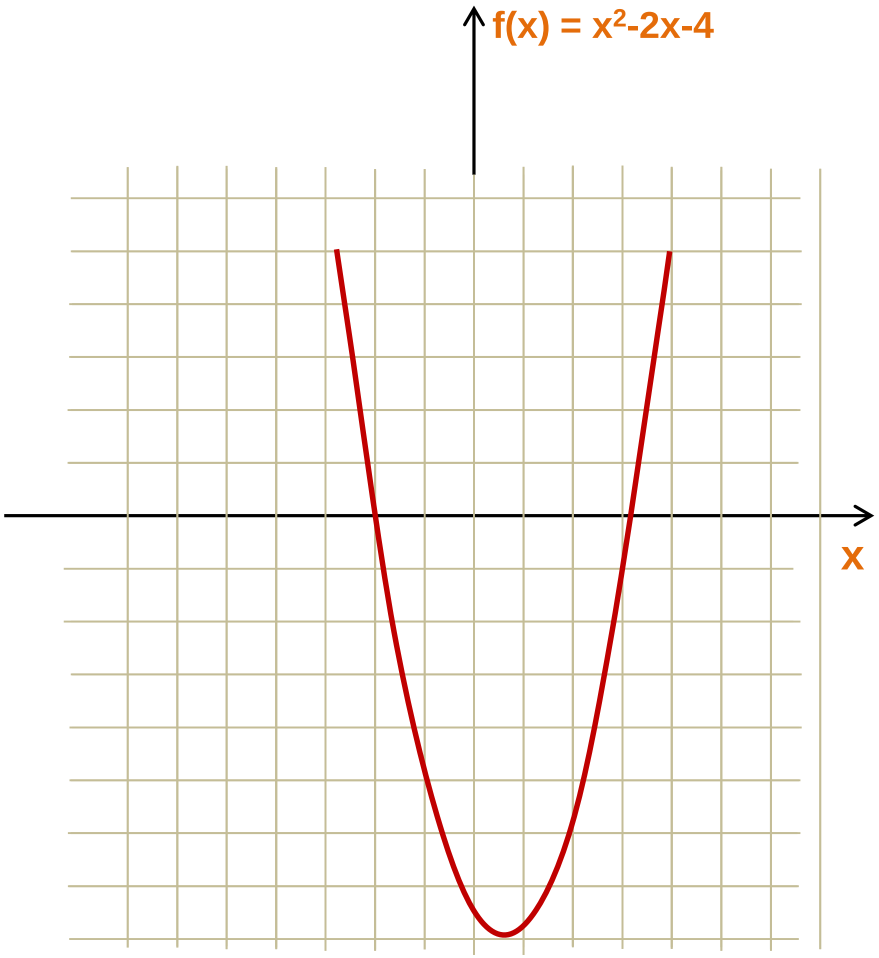


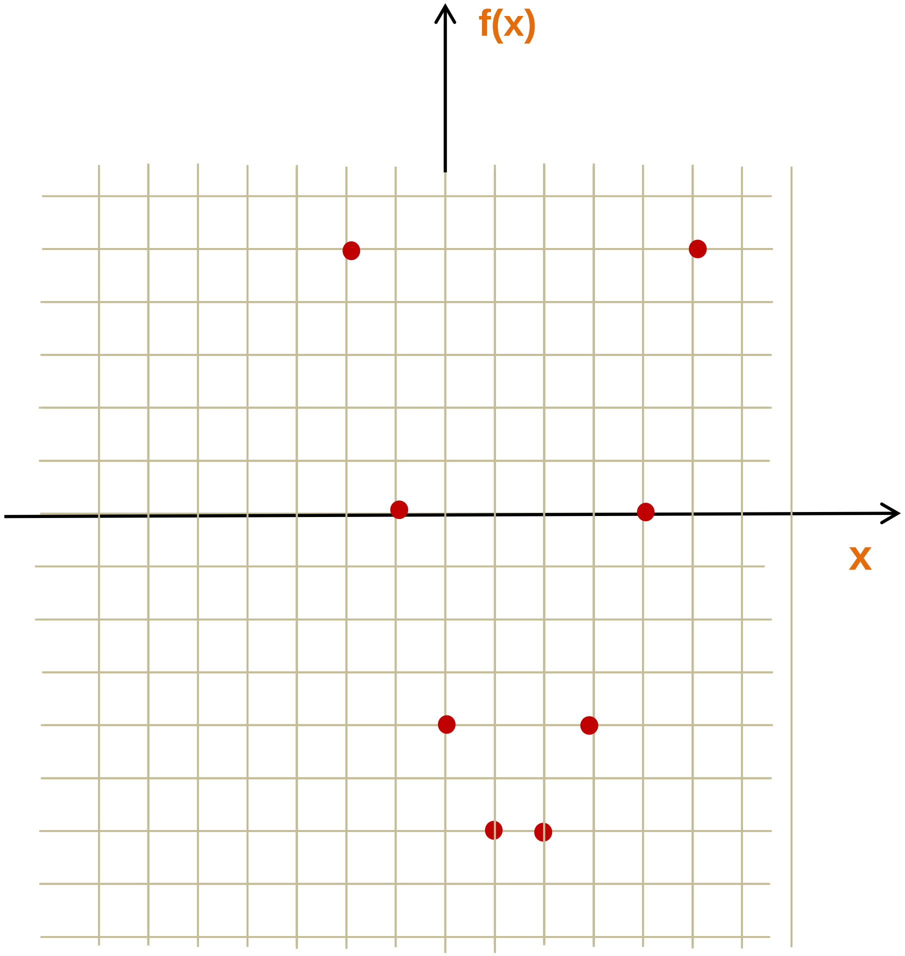
The set of integers is *discrete*.

We are mainly concerned with sets of objects that are either: (i) Finite, or; (ii) There exist a One-to-one correspondence with the integers.

0	**1**	**00**	**01**	**10**	**11**	**000**	**001**	**010**	**011**	**110**
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
1	2	3	4	5	6	7	8	9	10	11…….

Continuous vs. Discrete Function





The Foundations: Logic and Poofs

- **Understand and Reason with statements**

"There exists an integer that is not the sum of two squares"

"For every positive integer n, the sum of the positive integers not exceeding n is $n(n+1)/2$"

Definition

Mathematical logic provides methods for reasoning, rules and techniques to determine whether a statement or argument is valid

- **Example of a statement**:

If *x* is an even integer, then $x + 1$ is an odd integer

  - One needs to reason about this statement to determine whether it is ***true*** or ***false***
  - Such a statement is called Theorem (defined later)
  - NOTE: The above statement is true under the condition that "x is an integer" is true

Propositional Logic

- A **proposition**, or a **statement**, is a declarative sentence that is either true or false, but not both

Examples:

- 2 is an even number (true)
- Toronto is not the capital of Canada (true)

The following are not propositions:

- What time is it? (not a declarative sentence)
- $x+1 = 2$ (neither true nor false)

Propositional Logic

- Statements that are not **Propositions**:

  - ❑ How are you doing? (Question)
  - ❑ Do your home work on time! (command)
  - ❑ Bummer! (exclamation)

- Propositional variables:

  - **Variables** (usually lower case letters) are used to represent propositions, e.g., *p*, *q*, *r*, *s*

  - Have value: A proposition has usually a **truth value**: T if it is a true proposition and F if it is false.

Propositional Logic

- A **proposition**, or a **statement**, is a declarative sentence that is either true or false, but not both

Question?

- 2 + 3 = 5
- 3 + 4 = 6
- 4 is a prime number
- It is raining today
- It will rain tomorrow
- Chocolate is the best flavor of ice cream

Logical Operations: Complement

Definition: Complement

Let *p* be a proposition. The negation of *p* is denoted as $\neg p$ (or ~*p*) and is the statement: "It is *not* the case that *p*" or "not *p*". The truth values of *p* and ~*p* are opposite.

- **Find the negations of:**
  - Today is Friday
  - At least 10 inches of rain fell today in Miami

- **Truth Table:**

Truth table shows the value of an expression for every possible combination of values for the individual propositions constituting the expression.

p	$\neg p$
T	F
F	T

Definition: Conjunction

Let p and q be propositions. The conjunction of p and q, written $p \wedge q$, is the statement formed by joining propositions p and q using the word "and".

The statement $p \wedge q$ is true if both p and q are true; otherwise $p \wedge q$ is false.

Truth Table:

Definition: Conjunction

Let p and q be propositions. The conjunction of p and q, written $p \land q$, is the statement formed by joining propositions p and q using the word "and".

The statement $p \land q$ is true if both p and q are true; otherwise $p \land q$ is false.

Truth Table:

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operations: Disjunction

Definition: Disjunction

Let p and q be statements. The disjunction of p and q, written $p \vee q$, is the statement formed by joining statements p and q using the word "or".

The statement $p \vee q$ is false when both p and q are false; otherwise it is true.

Truth Table:

Definition: Disjunction

Let *p* and *q* be statements. The disjunction of *p* and *q*, written *p* V *q*, is the statement formed by joining statements *p* and *q* using the word "or".

The statement *p* V *q* is false when both *p* and *q* are false; otherwise it is true.

Truth Table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operations: Examples

Example 1:

- *p* is the statement “Today is Friday” *q* is the statement “It is raining today”, then the conjunction of *p* and *q* is:

“Today is Friday and it is raining today”

T or F? Statement is true only on rainy Fridays.

Example 2:

- The disjunction of *p* and *q* is:

“Today is Friday or it is raining today”

T or F? This proposition is T on any day that is either Friday or a rainy day (including rainy Fridays); it is F when it is not Friday and it does not rain.

Logical Operations: Exclusive OR

Definition: Exclusive OR

Let p and q be statements. The exclusive or of p and q, written $p \oplus q$, is the statement that is true when exactly one of p and q is true and false otherwise.

Truth Table:

TABLE 4 **The Truth Table for the Exclusive Or of Two Propositions.**

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Inclusive OR vs Exclusive OR

- Inclusive OR: (True if either or both propositions are true)
  - ❑ Do not take a medication if you have diabetes or you have a history of migraines.
- Exclusive OR: (True if exactly one of the propositions is true. False if both propositions are true.)
  - ❑ I will walk or drive to work.
- In logic: “OR” is always inclusive OR unless explicitly stated otherwise

Compound Propositions

Definition

- Symbols p ,q ,r ,...,called statement variables
- Symbols ¬, ∧, and v are called logical connectives

- Precedence of logical connectives is:
  - ¬ highest
  - ∧ second highest
  - v third highest

- **Examples**

  $\neg p \vee q$ is $(\neg p) \vee q$ and NOT $\neg (p \vee q)$

  $p \wedge q \vee r$ is $(p \wedge q) \vee r$ and not $p \wedge (q \vee r)$

Compound Propositions

- p: True; q: False; r: True

- $\neg p \vee (q \wedge \neg r)$
- $\neg (p \vee q \vee \neg r)$

Compound Propositions

- Complete the Truth table for $\neg p \vee (q \wedge \neg r)$

p	q	r	$\neg p$	$\neg r$	$q \wedge \neg r$	$\neg p \vee (q \wedge \neg r)$

Logical Equivalence

Definition

Two compound propositions are logically equivalent if they have the same truth value for every combination of truth values for their individual propositions

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$

De Morgan's Law:

De Morgan's Law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

- p: The applicant is over 18 years old
- q: The applicant has a valid driver's license

- $\neg (p \wedge q)$: It is not true the applicant is over 18 years of old and has a valid driver license.

- $\neg p \vee \neg q$: The applicant is not over 18 years old or does not have a valid driver's license.

De Morgan's Law:

De Morgan's Law: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

- p: The patient has a history of migraines
- q: The patient has diabetes

- $\neg (p \vee q)$: It is not true the patient has a history of migraines or has diabetes.

- $\neg p \wedge \neg q$: The patient does not have a history of migraines and does not have diabetes.

Logical Equivalence

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalence

- Show that $\neg (p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Definition

Let p and q be propositions. The statement "if p then q" is called an implication or a conditional statement and is denoted by $p \to q$.

It is false when (p is true and q is false) and true otherwise.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Operation

- *p* is called the hypothesis (or premise), *q* is called the conclusion (or consequence)

$p \to q$ is read:

If *p*, then *q*

p only if q

q unless $\neg p$

q whenever *p*

p implies *q*

Propositional Logic: Examples

- **Example:**
  - Let p: Today is Sunday, and;
    q: I will wash the car.
  - The implication $p \rightarrow q$ is the statement:
    - $p \rightarrow q$: If today is Sunday, then I will wash the car
  - If Juan has a smart phone, then 2+3 = 5
    - Conclusion is true → statement is always true
  - If Juan has a smart phone, then 2+3 = 6
    - True or False?
    - If Juan "has" a smart phone: F
    - If Juan "does not have" a smart phone: T

Conditional Operations: Examples

- **Example:**

  - Let p: Today is Sunday and q: I will wash the car.
    - $p \rightarrow q$: If today is Sunday, then I will wash the car
  - The converse of this implication is written $q \rightarrow p$
    - If I wash the car, then today is Sunday
  - The inverse of this implication is $\neg p \rightarrow \neg q$
    - If today is not Sunday, then I will not wash the car
  - The contrapositive of this implication is $\neg q \rightarrow \neg p$
    - If I do not wash the car, then today is not Sunday

Conditional Operation: Equivalent

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$

- p: Arash studied for his test
- q: Arash passed his test

- $p \rightarrow q$: If Arash studied for his test then he passed the test.

- $\neg p \vee q$: Arash did not study for his test or he passed the test.

Conditional Operation: Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$

- p: Arash studied for his test
- q: Arash passed his test

- $p \rightarrow q$: If Arash studied for his test then he passed the test.

- $\neg q \rightarrow \neg p$: If Arash did not pass his test, then he did not study for the test.

Compound Propositions (statement formula)

- Truth table of the compound proposition
  $(p \lor \neg q) \rightarrow (p \land q)$

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \rightarrow (p \land q)$

Bi-conditional Operation

Definition

Let p and q be statements. The statement "p if and only if q" is called the **bi-implication** or **bi-conditional** statement of p and q

- $p \leftrightarrow q$ is read:
  - "p is necessary and sufficient for q"
  - "q iff p"
  - "q when and only when p"

Bi-conditional Operation

Definition

Let p and q be statements. The statement "p if and only if q" is called the **bi-implication** or **bi-conditional** statement of p and q

– **Example:** p: You can take the flight,
q: You buy a ticket

$p \leftrightarrow q$:

You can take the flight iff you buy a ticket.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Bi-conditional Equivalent

$p \leftrightarrow q$ has the same truth table as: $(p \rightarrow q) \land (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \land (p \rightarrow q)$

Compound Propositions

Definition

- Symbols p ,q ,r ,...,called <u>statement variables</u>
- Symbols ¬, ∧, v, →,and ↔ are called <u>logical connectives</u>

- Precedence of logical connectives is:
  - ¬ highest
  - ∧ second highest
  - v third highest
  - → fourth highest
  - ↔ fifth highest
- **Examples**

  $\neg p \vee q$ is $(\neg p) \vee q$ and NOT $\neg (p \vee q)$

  $p \wedge q \vee r$ is $(p \wedge q) \vee r$ and not $p \wedge (q \vee r)$

Logical Equivalence

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TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
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$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Implication

Translation of $p \rightarrow q$	Example
if p, then q	If you cook dinner, then I'll take out the garbage.
if p, q	If you cook dinner, I'll take out the garbage.
p implies q	You cooking dinner implies I'll take out the garbage.
p only if q	You will cook dinner only if I take out the garbage.
p is sufficient for q	You cooking dinner is sufficient for me to take out the garbage.
q, if p	I'll take out the garbage if you cook dinner.
q when p	I'll take out the garbage when you cook dinner.
q whenever p	I'll take out the garbage whenever you cook dinner.
q is necessary for p	My taking out the garbage is necessary for you to cook dinner.
q unless $\neg$p	I'll take out the garbage unless you don't cook dinner.
q follows from p	My taking out the garbage follows from you cooking dinner.

- **Translating English sentences into expressions:**

Example:

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old"

Define the following statements:

p: You can ride the roller coaster

q: You are under 4 feet tall

r: You are older than 16 years old

Then the sentence can also be written as

"You cannot ride the roller coaster unless you are older than 16 years old OR NOT under 4 feet tall"

$$\neg (r \lor \neg q) \rightarrow \neg p$$

$$q \land \neg r \rightarrow \neg p$$

- **Translating English sentences into expressions:**

Example:

"You can access the Internet from campus <u>only if</u> you are an Electrical Engineering major or you are not freshman"

Define the following statements:

p: You are Electrical Engineering major

q: You are a freshman

r: You can access the Internet from campus

Then the sentence can be written as

"If you can access the Internet from campus, then you are a computer science major or you are not a freshman"

$$r \rightarrow p \vee \neg q$$

Definition

– A compound proposition is said to be a **tautology** if its truth value is always T for any assignment of the truth values to the statement variables occurring in *it*

– Show that $(p \land q) \rightarrow (p \lor q)$ is a tautology

Definition

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**

Determine which one of the compound propositions *)* is satisfiable

- $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$,
- $(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$,
- $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

What’s Next?

What’s Next:

- Get course materials ready
- Check the following materials on the course website
  - ❑ Course Summary
  - ❑ Hand out One (HO-1)

Topics of the Next Class

- The Foundations: Logic and Poofs
- Hand out Two (HO-2)