Version A

Calculus for the Life Sciences I

MAT1330 B Instructor: Elizabeth Maltais

Wednesday, October 4, 2017 Test #1

Duration: 75 minutes

FAMILY NAME:	SOLUTION
FIRST NAME:	
STUDENT NUMBER:	
†SIGNATURE:	

† **By signing above, you acknowledge that you have carefully read, understand, and will comply with the following instructions.**

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INSTRUCTIONS

- You have 75 minutes to complete this exam.
- This is a closed-book exam. Except for Faculty-approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no notes, cell phones, smartwatches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag under your desk for the duration of the exam**.
- Read each question carefully.
- Questions 1 through 4 are multiple choice, worth 1 point each. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 5 through 9 are short answer, with number of points as indicated. **You must show your work, your work must be legible, and you must record your answers in the boxes provided.**
- Where it is possible to check your work, do so.
- Good luck!

Marker's use only:

Question	Marks
1–4(/4)	
5, 6 (/4)	
7 (/7)	
8 (/5)	
9 (/4)	
Total (/24)	

55% remains $d=50$

1. (1 point) Suppose that a patient receives a daily dose of 50 mg/L of a certain drug such that 45% of it is eliminated from the body each day. If on a certain Monday, the concentration of the drug measured in their body (shortly after the daily dose) is 42 mg/L, which of the following Discrete-Time Dynamical Systems describe the dynamics of the concentration x_t of the drug in the body (in mg/L, t days after that Monday)?

$x_0 = 42$ mg/L

A. $x_{t+1} = .45x_t + 42$, with $x_0 = 50$

B. $x_{t+1} = .55x_t + 42$, with $x_0 = 50$

C. $x_{t+1} = .50x_t + 45$, with $x_0 = 42$

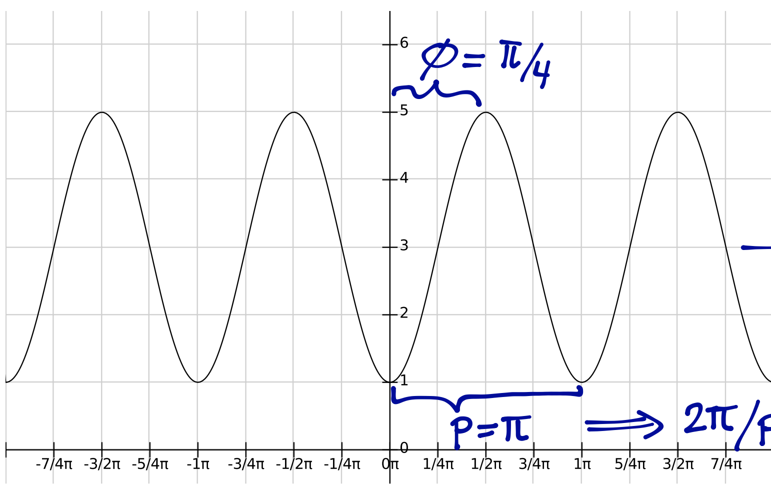
D. $x_{t+1} = .42x_t + 45$, with $x_0 = 50$

E. $x_{t+1} = .55x_t + 50$, with $x_0 = 42$

F. $x_{t+1} = 42x_t + 50$, with $x_0 = 45$

Your answer: E

2. (1 point) The following is the graph of a function $y = f(x)$.



$\phi = \pi/4$

$A=2$

$M=3$

$P=\pi \Longrightarrow 2\pi/P = 2$

Which of the following is a formula for $f(x)$?

A. $f(x) = 3 + 2\cos\left(2(x - \frac{\pi}{2})\right)$

B. $f(x) = 2 + 3\cos\left(2(x - \frac{\pi}{4})\right)$

C. $f(x) = 1 + 3\cos\left(4(x - \frac{\pi}{2})\right)$

D. $f(x) = 3 + 3\cos\left(4(x - \frac{\pi}{2})\right)$

E. $f(x) = 2 + \cos\left(x - \frac{\pi}{2}\right)$

F. $f(x) = 3 + \cos\left(x - \frac{\pi}{4}\right)$

Your answer: A

3. (1 point) Find the set of all solutions to the inequality

$$5 - \frac{6}{x} < 4.$$

A. $(-\infty, 6)$
B. $(0, 6)$
C. $(6, \infty)$
D. $(0, \infty)$
E. $(-\infty, 0) \cup (6, \infty)$
F. $(-\infty, 0)$

Your answer: B

$5-\frac{6}{x}<4 \Rightarrow 1-\frac{6}{x}<0 \Rightarrow \frac{x-6}{x}<0$

before multiplying by x we need to know its sign (to flip the ineq. or not)

Case 1 $x>0$

$\Rightarrow \frac{x-6}{x}<0 \Rightarrow x-6<0$
$\Rightarrow x<6$
$\Rightarrow 0<x<6$

Case 2 $x<0$

$\Rightarrow \frac{x-6}{x}<0 \Rightarrow x-6>0$
$\Rightarrow x>6$

Not possible for $x<0$ and $x>6$ so this case has no solutions

$\therefore$ solution set is $(0,6)$

4. (1 point) Find all x for which the following equality holds:

$$|8 - 3x^2| = 4.$$

A. $x = 0$
B. $x = 2/\sqrt{3}$
C. $x = \pm 2/\sqrt{3}$
D. $x = \pm 2$ or $x = \pm 2/\sqrt{3}$
E. $x = \pm\sqrt{8/3}$
F. $x = 2 \pm \sqrt{8/3}$

Your answer: D

Case 1. $8-3x^2 \geq 0$
$8 \geq 3x^2$
$\frac{8}{3} \geq x^2$
$\sqrt{8/3} \geq |x|$
$\Rightarrow -\sqrt{\frac{8}{3}} \leq x \leq \sqrt{\frac{8}{3}}$

Equality becomes
$|8-3x^2|=4$
so $8-3x^2=4$
$\Rightarrow 4=3x^2$
$\Rightarrow x=\pm\sqrt{\frac{4}{3}}$
$=\pm 2/\sqrt{3}$

Case 2. $8-3x^2<0$
$8<3x^2$
$\frac{8}{3}<x^2$
$\sqrt{8/3}<|x|$
$\Rightarrow x<-\sqrt{8/3}$ or $x>\sqrt{\frac{8}{3}}$

Equality becomes
$|8-3x^2|=4$
so $-(8-3x^2)=4 \Rightarrow 8-3x^2=-4$
$\Rightarrow 12=3x^2$
$\Rightarrow x=\pm 2$

5. (2 points) Find all solutions x of the following equation. Show your work.

$$\ln(x+2) + \ln(x-3) = \ln(x+9)$$

Your work:

$\ln((x+2)(x-3)) = \ln(x+9)$

$\Rightarrow \ln(x^2-x-6) = \ln(x+9)$

$\Rightarrow e^{\ln(x^2-x-6)} = e^{\ln(x+9)}$

$\Rightarrow x^2-x-6 = x+9$

$\Rightarrow x^2-2x-15=0$

$\Rightarrow (x+3)(x-5)=0$

$\cancel{x=-3}$ $\quad x=5$

reject because, from the original eq^{n}, x must be in the domains of $\ln(x+2)$, $\ln(x-3)$ and $\ln(x+9)$ $\therefore x>3$

Your answer: $\boxed{x=5}$

6. (2 points) Let $f(x)$ be the following piecewise-defined function, where k is a constant real number.

$$f(x) = \begin{cases} k\cos(\pi x) & \text{if } x < 5 \\ \frac{x}{2} - 4 & \text{if } x \geq 5. \end{cases}$$

For which value of k does the following limit exist? You must justify your answer by explaining what is needed for the limit to exist and showing how you solved for k.

$$\lim_{x\to 5} f(x)$$

Your work: we need $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$

left-side limit

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} k\cos(\pi x)$

$= k\cos(5\pi)$

$= k(-1)$

$= -k$

right-side limit

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{x}{2} - 4$

$= \frac{5}{2} - 4$

$k = \boxed{3/2}$

$\therefore$ for limit to exist, we need $-k = \frac{5}{2} - 4 \Rightarrow k = 3/2 = 1.5$

7. (7 points) The population of fish in Fisher's Pond grow at a steady rate annually, but fishing is so popular that, without restocking, the population would die out. Therefore Fisher's Pond is restocked with fish each spring. Using historical data, we declare that DTDS modeling the population of fish, with x_t representing the average number of fish per m^2 of surface area in year t, is given by

$$\boxed{x_{t+1} = 0.9x_t + 4.5}$$

(a) (1 point) Give the updating function f for this DTDS. $f(x) = \boxed{0.9x + 4.5}$

(b) (1 point) Find the fixed point x^* of this DTDS.

for linear $f(x) = rx + d$, $r \neq 1$, $x^* = \frac{d}{1-r} = \frac{4.5}{1-0.9} = 45$

$x^* = \boxed{45}$

Alternatively, for any DTDS, fixed points are any solutions to $x = f(x)$
$\Rightarrow x = 0.9x + 4.5 \Rightarrow x - 0.9x = 4.5 \Rightarrow 0.1x = 4.5 \Rightarrow x = 45$

(c) (1 point) Suppose that in year zero there were 15 fish/m^2. Give the general solution formula to this DTDS.

for $f(x) = rx + d$ with initial condition x_0, the general solution formula is $x_t = r^t(x_0 - x^*) + x^*$ where $x^* = \frac{d}{1-r}$

General solution formula: $\boxed{x_t = (0.9)^t(15 - 45) + 45}$

In this DTDS: $r = 0.9$
$d = 4.5$
$x_0 = 15$
$x^* = 45$

(d) (1 point) Find the number of fish per m^2 after two years.

$x_2 = (0.9)^2(15 - 45) + 45 = 20.7$

Alternatively, $x_1 = f(x_0) = 0.9(15) + 4.5 = 18$
$x_2 = f(x_1) = 0.9(18) + 4.5 = 20.7$

Your answer: $\boxed{x_2 = 20.7 \text{ fish}/m^2}$

answer does not need to be rounded because x_t measures density of fish (#fish)/(m^2) which need not be an integer

(e) (3 points) Determine the (whole) number of years necessary until the number of fish per m^2 is within 0.3 of the fixed point, that is, until $|x_t - x^*| < 0.3$. **Show your work. Your answer must be clear and well-justified to earn full marks.**

We need to find all solutions to the inequality $|x_t - x^*| < 0.3$

$\Rightarrow \left|\underbrace{0.9^t(15-45) + 45}_{x_t} \underbrace{-45}_{x^*}\right| < 0.3$

$\Rightarrow |0.9^t(-30)| < 0.3$

$\Rightarrow 0.9^t(30) < 0.3$

(since $t \geq 0$, $0.9^t(30) \geq 0$)

$\Rightarrow 0.9^t < \frac{0.3}{30}$

$\ln(0.9^t) < \ln\left(\frac{0.3}{30}\right)$

$\Rightarrow t\ln(0.9) < \ln(0.3/30)$

$\Rightarrow t > \frac{\ln(0.3/30)}{\ln(0.9)}$

($\ln(0.9)$ is negative because $0.9 < 1$ ∴ ineq. sign flips)

$\Rightarrow t > 43.70\ldots$ years
$\Rightarrow t \geq 44$ years

8. (5 points) The DTDS $x_{t+1} = 0.5x_t(5.1 - x_t)$ models a certain population.

(a) (2 points) Solve for all fixed points of the DTDS.

Your work: fixed points are solutions to $x = f(x)$.

In this DTDS, $f(x) = 0.5x(5.1-x)$

So we solve $x = 0.5x(5.1-x)$

$\Rightarrow x = (0.5)(5.1)x - 0.5x^2$

$\Rightarrow 0 = (0.5)(5.1)x - 0.5x^2 - x$

$\Rightarrow 0 = 1.55x - 0.5x^2$

$\Rightarrow 0 = x[1.55 - 0.5x]$

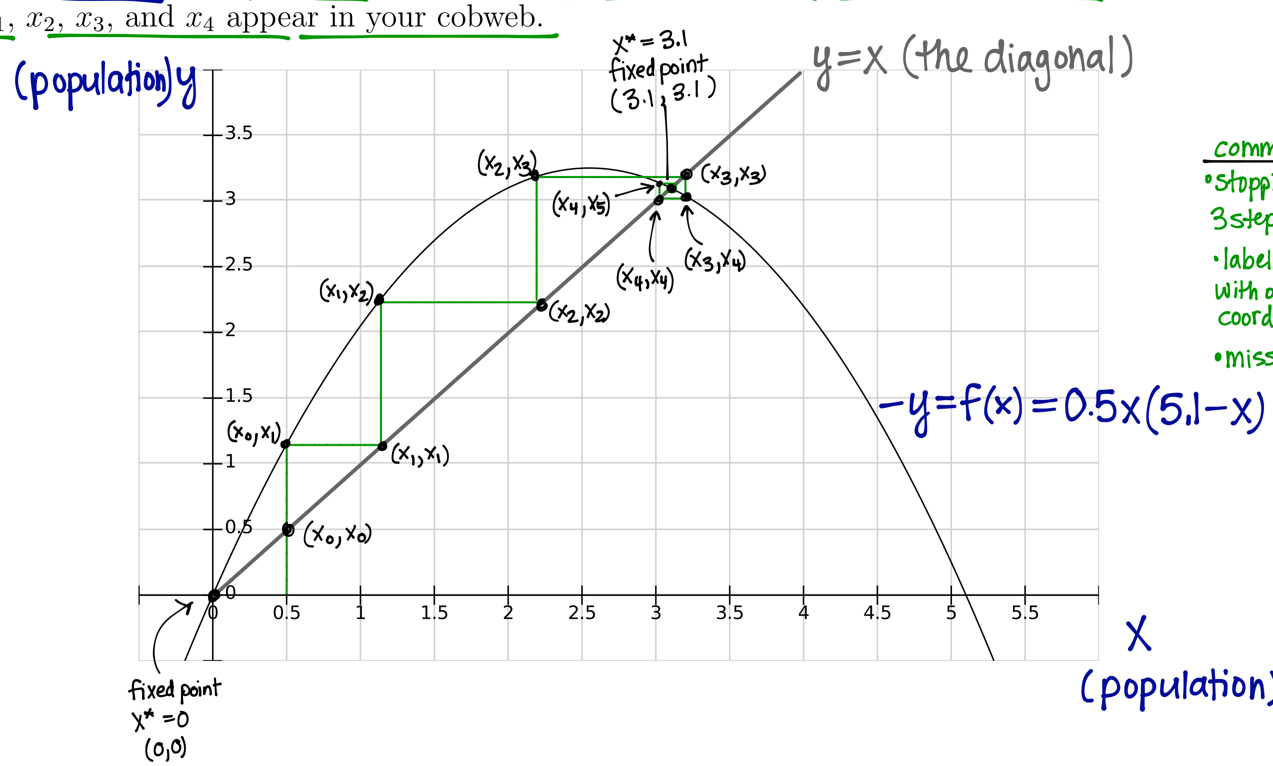
$x = 0 \qquad x = \frac{1.55}{0.5} = 3.1$

common mistakes:

- cancelling x from both sides (only works if $x \neq 0$) so you "lose" the second fixed point $x^* = 0$.
- solving $0 = f(x)$ instead of $x = f(x)$.
- trying to use $x^* = \frac{d}{1-r}$ formula when the updating function is not of the form $f(x) = rx + d$.

Your answer: $\boxed{x^* = 0 \text{ and } x^* = 3.1}$

(b) (2 points) The graph of the updating function of this DTDS is given below. Suppose the initial value is $x_0 = 0.5$. Draw a cobweb diagram on the graph below for this DTDS with at least $\boxed{\text{4 steps.}}$ Label the axes, the functions, the fixed points, and clearly indicate wherever x_0, x_1, x_2, x_3, and x_4 appear in your cobweb.



common mistakes

- stopping after 3 steps.
- labelling points with only one coordinate
- missing labels

(c) (1 point) Write a sentence to explain what happens in the long term if $x_0 = 0.5$. Your sentence should include the word "stable" or "unstable", as well as the exact fixed point from (a) which is relevant.

If $x_0 = 0.5$, then, in the long term, the solution $\{x_0, x_1, x_2, \ldots\}$ approaches the fixed point $x^* = 3.1$ which is a stable fixed point

common mistakes (which were mostly generously forgiven)

- describing anything other than a fixed point as being stable (or unstable)
- saying $x^* = 3.1$ is stable because at least one solution approaches it

No! for a fixed point to be stable, all nearby solutions must approach (or at least stay near) the fixed point

9. (4 points) Decide if the following limits exists. For each one, if it exists, evaluate the limit exactly using algebraic methods, showing all your steps. If it does not exist, justify your answer clearly using mathematical reasoning.

(a) (2 points) $\lim_{x \to -3} \dfrac{x+3}{4-\sqrt{25-x^2}}$

$$= \lim_{x\to-3}\left(\frac{x+3}{4-\sqrt{25-x^2}}\right)\left(\frac{4+\sqrt{25-x^2}}{4+\sqrt{25-x^2}}\right)$$

$$= \lim_{x\to-3}\frac{(x+3)(4+\sqrt{25-x^2})}{4^2+4\sqrt{25-x^2}-4\sqrt{25-x^2}-(\sqrt{25-x^2})^2}$$

$$= \lim_{x\to-3}\frac{(x+3)(4+\sqrt{25-x^2})}{16-(25-x^2)}$$

$$= \lim_{x\to-3}\frac{(x+3)(4+\sqrt{25-x^2})}{-9+x^2}$$

$$= \lim_{x\to-3}\frac{(x+3)(4+\sqrt{25-x^2})}{x^2-9}$$

$$= \lim_{x\to-3}\frac{(x+3)(4+\sqrt{25-x^2})}{(x-3)(x+3)}$$

$$= \lim_{x\to-3}\frac{\cancel{(x+3)}(4+\sqrt{25-x^2})}{(x-3)\cancel{(x+3)}}$$

$$= \lim_{x\to-3}\frac{4+\sqrt{25-x^2}}{x-3}$$

$$= \frac{4+\sqrt{25-(-3)^2}}{-3-3}$$

$$= \frac{4+\sqrt{16}}{-6} = \frac{4+4}{-6} = -\frac{8}{6} = -\frac{4}{3}$$

Your answer: $\boxed{\lim_{x\to-3}\dfrac{x+3}{4-\sqrt{25-x^2}} = -\dfrac{4}{3}}$

* see next page for common mistakes!

(b) (2 points) $\lim_{x \to \infty} \dfrac{3x^3}{\sqrt{5x^4+3}}$

$$= \lim_{x\to\infty}\left(\frac{3x^3}{\sqrt{5x^4+3}}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$$ (divide numerator and denom. both by same highest power of x in the denom.)

$$= \lim_{x\to\infty}\frac{\frac{3x^3}{x^2}}{\frac{\sqrt{5x^4+3}}{x^2}} = \lim_{x\to\infty}\frac{3x}{\frac{\sqrt{5x^4+3}}{\sqrt{x^4}}}$$ because $x\to\infty$, $x>0$, so $x^2=\sqrt{x^4}$

$$= \lim_{x\to\infty}\frac{3x}{\sqrt{\frac{5x^4+3}{x^4}}} = \lim_{x\to\infty}\frac{3x}{\sqrt{5+\frac{3}{x^4}}} = \lim_{x\to\infty}\frac{3x}{\sqrt{5+0}} = \infty$$

* see next page for common mistakes!

Your answer: $\boxed{\lim_{x\to\infty}\dfrac{3x^3}{\sqrt{5x^4+3}} = \infty}$ (DNE)

Common mistakes (for Q9a)

- Simplifying $\left(4-\sqrt{25-x^2}\right)\left(4+\sqrt{25-x^2}\right)$ incorrectly:

$*$ many wrote $\left(4-\sqrt{25-x^2}\right)\left(4+\sqrt{25-x^2}\right) = 16-25-x^2$

some wrote $\left(4-\sqrt{25-x^2}\right)\left(4+\sqrt{25-x^2}\right) = 16+25-x^2$

Only $\left(4-\sqrt{25-x^2}\right)\left(4+\sqrt{25-x^2}\right) = 16-(25-x^2) = 16-25+x^2$ is correct.

$*$ if you simplified (incorrectly) that $\left(4-\sqrt{25-x^2}\right)\left(4+\sqrt{25-x^2}\right) = 16-25-x^2$

then many of you made a second mistake: $16-25-x^2 = -(x^2+9)$ (true)

but $\neq -(x-3)(x+3)$

(x^2+9) does not factor.

Common mistakes (for Q9b)

- not dividing numerator and denom. by the same power of x.
- not converting the power of x from within/out of square root as being $\sqrt{x^4} = x^2$
- claiming $\sqrt{5x^4+3} = \sqrt{5x^4}+\sqrt{3}$ (very wrong!)
- if rationalizing first (which is ok but unnecessary for this limit) limit becomes

$$\lim_{x\to\infty} \frac{(3x^3)\sqrt{5x^4+3}}{5x^4+3}$$

many of you then divided by x^4 (which is the right idea) but you actually divided num./denom. by different powers of x:

$$\lim_{x\to\infty} \frac{(3x^3)\left(\sqrt{4x^4+3}\right)}{5x^4+3} \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} \quad ✓$$

$$\times \quad \lim_{x\to\infty} \frac{\dfrac{3x^3}{x^4}\cdot\dfrac{\sqrt{4x^4+3}}{x^4}}{\dfrac{5x^4+3}{x^4}}$$

dividing by x^4 twice in numerator means you are actually dividing the numerator by $\underline{x^8}$ not $\underline{x^4}$