CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Math 205: Differential and Integral Calculus 2 **Practice Problems**

Note: The problems are not meant to represent the format nor the difficulty of an actual exam. They are for study purposes only.

1. (a) Derive the sigma notation formula for the right Riemann sum R_n of the function $f(x) = 1 + 3x^2$ on the interval $[0, 2]$ using n intervals of equal length.

   (b) Calculate the integral $\int_0^2 f(x)\,dx$ by evaluating the limit of R_n as $n \to \infty$.

$$\text{Reminder: } \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. (a) Sketch the graph of $g(x) = 1 + \sqrt{1 - (x-1)^2}$.

   (b) Use part (a) of this item to evaluate the integral of $\int_0^2 g(x)\,dx$ in terms of area.

3. Use the Fundamental Theorem of Calculus to find $F'(x)$ if

$$F(x) = \int_{x^2-1}^{x} e^{t^2}\,dt$$

4. Calculate the following indefinite integrals.

   (a) $\displaystyle\int \frac{x^3}{\sqrt{x^2-9}}\,dx$ (b) $\displaystyle\int \frac{1-4x}{2x^2-3x+1}\,dx$

5. Evaluate the following definite integrals.

   (a) $\displaystyle\int_1^e x \ln x\,dx$ (b) $\displaystyle\int_0^{\pi/4} \cos^5 x \tan^2 x\,dx$

6. Let R be the region enclosed by the curves $y = |x| - 1$ and $y = 1 - x^2$.

   (a) Find the area of R using the variable x.

   (b) Find the area of R using the variable y.

   (c) Find the volume of the solid obtained when R is revolved about the line $y = 2$.

Midterm Practice problems solution.

(1) (a) $f(x) = 1 + 3x^2$, $x \in [0,2]$.

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

$$x_i^* = x_i = 0 + i\Delta x = \frac{2i}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \sum_{i=1}^{n}\left[1 + 3\left(\frac{2i}{n}\right)^2\right]\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n}\frac{2}{n} + \sum_{i=1}^{n}\frac{24i^2}{n^3}$$

$$= 2 + \frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

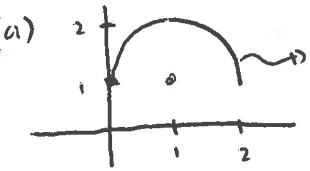
$$= \boxed{2 + 4\,\frac{n(n+1)(2n+1)}{n^3}}$$

(b) $$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty}\left(2 + \frac{4n(n+1)(2n+1)}{n^3}\right)$$

$$= \boxed{10}$$

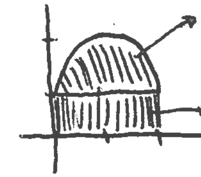
(2) (a)



semicircle.

$y = 1 + \sqrt{1-(x-1)^2}$

(b) $g(x) = 1 + \sqrt{1-(x-1)^2}$



semicircle area $= \frac{\pi(1)^2}{2}$.

rectangle area $= 2\cdot 1$

$$\int_0^2 g(x)\,dx = \boxed{2 + \frac{\pi(1)^2}{2}}$$

(3.) $$F(x) = \int_{x^2-1}^{x} e^{t^2}dt = \int_{x^2-1}^{0} e^{t^2}dt + \int_0^x e^{t^2}dt$$

$$= -\int_0^{x^2-1} e^{t^2}dt + \int_0^x e^{t^2}dt$$

$$\therefore \boxed{F'(x) = -e^{(x^2-1)^2}\cdot 2x + e^{x^2}}$$

(4.) (a) $$\int \frac{x^3}{\sqrt{x^2-9}}\,dx = \frac{1}{2}\int\frac{u+9}{\sqrt{u}}\,du = \frac{1}{2}\int\left(u^{1/2} + 9u^{-1/2}\right)du$$

Let $u = x^2 - 9 \Rightarrow x^2 = u + 9$

$du = 2x\,dx$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2} + \frac{9u^{1/2}}{1/2}\right) + C$$

$$= \boxed{\frac{(x^2-9)^{3/2}}{3} + 9(x^2-9)^{1/2} + C}$$

(4)(b) $\displaystyle\int \frac{1-4x}{2x^2-3x+1}\,dx = \int \frac{1-4x}{(2x-1)(x-1)}\,dx$

$$\frac{1-4x}{(2x-1)(x-1)} = \frac{A}{2x-1} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(2x-1)}{(2x-1)(x-1)}$$

$$A(x-1)+B(2x-1) = 1-4x$$

$x=1 \Rightarrow B(2\cdot 1-1) = 1-4\cdot 1$

$B=-3$

$x=\frac{1}{2} \Rightarrow A\left(\frac{1}{2}-1\right) = 1-4\cdot\frac{1}{2}$

$-\frac{A}{2} = -1$

$A = 2$

$$\therefore \int \frac{1-4x}{2x^2-3x+1}\,dx = \int\left(\frac{2}{2x-1} - \frac{3}{x-1}\right)dx$$

$$= \frac{2\ln|2x-1|}{2} - 3\ln|x-1| + C$$

$$= \boxed{\ln|2x-1| - 3\ln|x-1| + C}$$

(5)(a) $\displaystyle\int_1^e x\ln x\,dx = \frac{x^2}{2}\ln x\Big|_1^e - \int_1^e \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx = \frac{x^2}{2}\ln x\Big|_1^e - \frac{x^2}{4}\Big|_1^e$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$

$$= \left(\frac{e^2}{2} - 0\right) - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \boxed{\frac{e^2+1}{4}}$$

(b) $\displaystyle\int_0^{\pi/4} \cos^5 x\tan^2 x\,dx = \int_0^{\pi/4} \cos^3 x\sin^2 x\,dx$

$$= \int_0^{\pi/4} \cos^2 x\sin^2 x\cos x\,dx$$

$$= \int_0^{\pi/4} (1-\sin^2 x)\sin^2 x\cos x\,dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int_{\square}^{\square} (1-u^2)u^2\,du$$

$$= \int_{\square}^{\square} (u^2-u^4)\,du$$

$$= \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_{\square}^{\square}$$

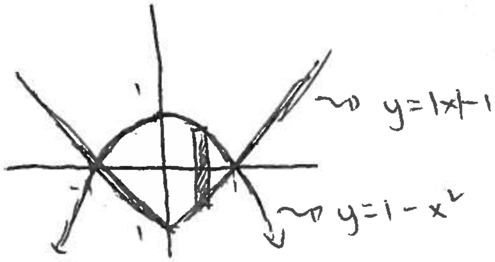
$$= \left(\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{3}\cdot\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 - 0$$

$$= \frac{1}{6\sqrt{2}} - \frac{1}{20\sqrt{2}}$$

$$= \boxed{\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{40}}$$

(6)



(a) $A=\int_{-1}^{1}\underbrace{[(1-x^2)-(|x|-1)]}_{\text{even function}}\,dx$

$=2\int_0^1 (1-x^2-|x|+1)\,dx$

$=2\int_0^1 (2-x^2-x)\,dx$

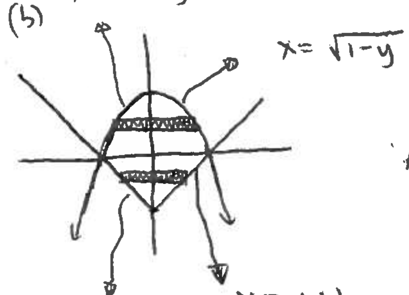
$=2\left(2x-\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_0^1$

$=2\left(2-\frac{1}{3}-\frac{1}{2}\right)$

$=2\left(\frac{7}{6}\right)$

$=\boxed{\frac{7}{3}}$

(b)



$A=\int_{-1}^{0}[(y+1)-(-(y+1))]\,dy$

$+\int_0^1\left(\sqrt{1-y}-(-\sqrt{1-y})\right)dy$

$=\int_{-1}^{0}(2y+2)\,dy+\int_0^1 2\sqrt{1-y}\,dy$

↳ $u=1-y$
$du=-dy$

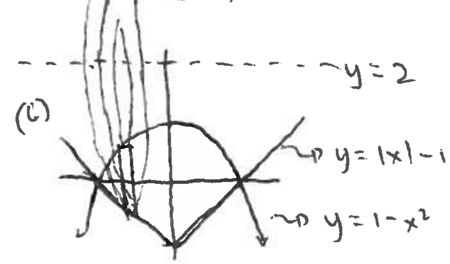
$=(y^2+2y)\Big|_{-1}^{0}+\frac{2(1-y)^{3/2}}{-3/2}\Big|_0^1$

$=0-(1-2)-\frac{4}{3}(0-1)$

$=1+\frac{4}{3}$

$=\boxed{\frac{7}{3}}$

(c)



$R=2-(|x|+1)=3-|x|$

$r=2-(1-x^2)=1+x^2$

$\therefore V=\int_{-1}^{1}\pi(R^2-r^2)\,dx$

$=\pi\int_{-1}^{1}\underbrace{[(3-|x|)^2-(1+x^2)^2]}_{\text{even function}}\,dx$

$=2\pi\int_0^1[(3-x)^2-(1+x^2)^2]\,dx$

$=2\pi\int_0^1[(9-6x+x^2)-(1+2x^2+x^4)]\,dx$

$=2\pi\left[9x-3x^2+\frac{x^3}{3}-x-\frac{2x^3}{3}-\frac{x^5}{5}\right]\Big|_0^1$

$=2\pi\left(9-3+\frac{1}{3}-1-\frac{2}{3}-\frac{1}{5}\right)=2\pi\left(5-\frac{1}{3}-\frac{1}{5}\right)=\boxed{\frac{134\pi}{15}}$