MATH 263
WEBWORK 5

1. $(x^2+1)y''-xy'+y=0$, $y(0)=a_0=6$, $y'(0)=a_1=3$

$y=\sum_{n=0}^{\infty} a_n x^n \Rightarrow y'=\sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$2a_2+a_0+6a_3x+\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_n x^n = 0$

$2a_2+a_0+6a_3x+\sum_{n=2}^{\infty}\left[n(n-1)a_n+(n+2)(n+1)a_{n+2}-na_n+a_n\right]x^n=0$

$(n+2)(n+1)a_{n+2} = -(n(n-1)a_n - (n-1)a_n)$

$a_{n+2} = \frac{-a_n(n-1)(n-1)}{(n+2)(n+1)}$

$a_{n+2} = \frac{-a_n(n-1)^2}{(n+2)(n+1)}$

$2a_2+a_0=0$

$2a_2=-6$

$a_2=-3$

$6a_3x=0$

$a_3=0$

$a_4=\frac{-a_2(1)^2}{(4)(3)}$

$a_4=\frac{+(\cancel{+3})}{(4)(\cancel{3})}$

$a_4=\frac{1}{4}$

$a_5=\frac{-a_3(2)^2}{(5)(4)}$

$a_5=\frac{-(0)(4)}{(5)(4)}$

$a_5=0$

$a_6=\frac{-a_4(3)^2}{(6)(5)}$

$a_6=\frac{-(1)(3)^2}{(4)(6)(5)}$

$a_6=\frac{-\cancel{9}3}{(20)(\cancel{6})2}$

$a_6=\frac{-3}{40}$

$a_7=\frac{-a_5(4)^2}{(7)(6)}$

$a_7=\frac{-(0)(16)}{(7)(6)}$

$a_7=0$

$a_8=\frac{-a_6(5)^2}{(8)(7)}$

$a_8=\frac{-(-3)(\cancel{25})5}{8\,(\cancel{40})(8)(7)}$

$a_8=\frac{15}{448}$

$\therefore\ 6+3x-3x^2+\frac{1}{4}x^4-\frac{3}{40}x^6+\frac{15}{448}x^8$

2. $(x^2-x+1)y''-y'+7y=0 \quad y(0)=0,\ y'(0)=-4$

$$y=\sum_{n=0}^{\infty}a_nx^n \Rightarrow y'=\sum_{n=1}^{\infty}na_nx^{n-1} \Rightarrow y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

$$(x^2-x+1)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-1}+7\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-1}+7\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+7\sum_{n=0}^{\infty}a_nx^n=0$$

$$-2a_2x+2a_2+6a_3x-a_1-2a_2x+7a_0+7a_1x+\sum_{n=2}^{\infty}\left[n(n-1)a_n-n(n+1)a_{n+1}+(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}+7a_n\right]x^n=0$$

$$n^2a_n-na_n-n^2a_{n+1}-na_{n+1}+n^2a_{n+2}+3na_{n+2}+2a_{n+2}-na_{n+1}-a_{n+1}+7a_n=0$$

$$n^2a_{n+2}+3na_{n+2}+2a_{n+2}-n^2a_{n+1}-2na_{n+1}-a_{n+1}+n^2a_n-na_n+7a_n=0$$

$$a_{n+2}(n^2+3n+2)-a_{n+1}(n^2+2n+1)+a_n(n^2-n+7)=0$$

$$a_{n+2}(n+1)(n+2)=a_{n+1}(n+1)^2-a_n(n^2-n+7)$$

$$a_{n+2}=\frac{a_{n+1}(n+1)^2-a_n(n^2-n+7)}{(n+1)(n+2)}$$

$a_0=0$

$a_1=-4$

$2a_2-a_1+7a_0=0$

$2a_2=-4$

$a_2=-2$

$-2a_2x+6a_3x-2a_2x+7a_1x=0$

$6a_3=4a_2-7a_1$

$a_3=\frac{4(-2)-7(-4)}{6}$

$a_3=\frac{-8+28}{6}$

$a_3=\frac{20}{6}$

$a_3=\frac{10}{3}$

$a_4=\frac{a_3(3)^2-a_2(4-2+7)}{(3)(4)}$

$a_4=\frac{9(10/3)-(-2)(9)}{12}$

$a_4=\frac{30+18}{12}$

$a_4=\frac{48}{12}$

$a_4=4$

$a_5=\frac{a_4(4)^2-a_3(9-3+7)}{(4)(5)}$

$a_5=\frac{(4)(16)-(10/3)(13)}{20}$

$a_5=\frac{64-130/3}{20}$

$a_5=\frac{\cancel{62}\ 31}{\cancel{320}\ 30}$

$a_5=\frac{31}{30}$

$$y=-4x-2x^2+\frac{10}{3}x^3+4x^4+\frac{31}{30}x^5$$

3. $y'' - (\sin x)y = \cos x \quad , \quad y(0) = 4 \ , \ y'(0) = 9$

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{1+2n}}{(1+2n)!} \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for x^0: $2a_2 = 1$

$a_2 = 1/2$

for x': $3(2)a_3 x - \left(\frac{x}{1}\right)(a_0) = 0$

$6a_3 - a_0 = 0$

$a_3 = \frac{a_0}{6}$

$a_3 = \frac{4}{6}$

$a_3 = \frac{2}{3}$

for x^2: $4(3)a_4 x^2 - \left(\frac{x}{1}\right)(a_1 x) = \frac{-x^2}{2}$

$12a_4 - a_1 = -\frac{1}{2}$

$12a_4 = 9 - \frac{1}{2}$

$12a_4 = \frac{17}{2}$

$a_4 = \frac{17}{24}$

$$\therefore y = 4 + 9x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{17}{24}x^4$$

4. $2x(x-1)y'' + 3(x-1)y' - y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \Rightarrow y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2}$$

$$2\sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-1} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r} - 2\sum_{n=-1}^{\infty} (n+r)(n+r+1) a_{n+1} x^{n+r} + 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - 3\sum_{n=-1}^{\infty} (n+r+1) a_{n+1} x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r-1)(n+r)a_n + 3(n+r)a_n - a_n] x^{n+r} + \sum_{n=-1}^{\infty} [-2(n+r)(n+r+1)a_{n+1} - 3(n+r+1)a_{n+1}] x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r-1)(n+r)a_n + 3(n+r)a_n - a_n] x^{n+r} - 2(r-1)(r)a_0 x^{n+r} - 3ra_0 x^{n+r} + \sum_{n=0}^{\infty} [-2(n+r)(n+r+1)a_{n+1} - 3(n+r+1)a_{n+1}] x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [2(n+r-1)(n+r)a_n + 3(n+r)a_n - 2(n+r)(n+r+1)a_{n+1} - 3(n+r+1)a_{n+1}] x^{n+r} = 0$$

$-2(r^2 - r)a_0 x^{n+r} - 3ra_0 x^{n+r}$

$$\sum_{n=0}^{\infty} [2(n+r-1)(n+r)a_n + 3(n+r)a_n - a_n - 2(n+r)(n+r+1)a_{n+1} - 3(n+r+1)a_{n+1}] x^{n+r}$$

$0 = (-2r^2 + 2r - 3r)a_0 x^{n+r}$

$0 = (-2r^2 - r)a_0 x^{n+r}$

$0 = -r(2r+1)$

$r_2 = 0 \quad r_1 = -1/2$

$\therefore$ indicial equation: $r^2 + \frac{1}{2}r = 0$

for $r=-1/2$, $a_0=7$

$$\sum_{n=0}^{\infty}\left[2(n-3/2)(n-1/2)a_n+3(n-1/2)a_n-a_n-2(n-1/2)(n+1/2)a_{n+1}-3(n+1/2)a_{n+1}\right]x^{n+r}=0$$

$$2(n^2-\tfrac{1}{2}n-\tfrac{3}{2}n+\tfrac{3}{4})a_n+3na_n-\tfrac{3}{2}a_n-a_n-2(n^2-1/4)a_{n+1}-3na_{n+1}-\tfrac{3}{2}a_{n+1}=0$$

$$2n^2a_n-4na_n+\tfrac{3}{2}a_n+3na_n-\tfrac{3}{2}a_n-a_n=2n^2a_{n+1}-\tfrac{1}{2}a_{n+1}+3na_{n+1}+\tfrac{3}{2}a_{n+1}$$

$$(2n^2-n-1)a_n=(2n^2+3n+1)a_{n+1}$$

$$a_{n+1}=\frac{(2n+1)(n-1)a_n}{(2n^2+3n+1)}$$

$a_1=\frac{(1)(-1)a_0}{(1)}$ $\quad a_2=\frac{(3)(0)a_1}{(6)}$ $\quad a_3=\frac{(5)(1)\cancel{a_2}^0}{(15)}$ $\quad a_4=\frac{(7)(2)\cancel{a_3}^0}{(28)}$

$a_1=-7$ $\quad a_2=(0)$ $\quad a_3=0$ $\quad a_4=0$

$\therefore$ a) $y=x^{r_1}(7-7x+0x^2+0x^3+0x^4+\dots)$

for $r=0$, $a_0=8$

$$\sum_{n=0}^{\infty}\left[2(n-1)na_n+3na_n-a_n-2n(n+1)a_{n+1}-3(n+1)a_{n+1}\right]x^{n+r}=0$$

$$2n^2a_n-2na_n+3na_n-a_n=2n^2a_{n+1}+2na_{n+1}+3na_{n+1}+3a_{n+1}$$

$$(2n^2+n-1)a_n=(2n^2+5n+3)a_{n+1}$$

$$a_{n+1}=\frac{\cancel{(n+1)}(2n-1)a_n}{(2n+3)\cancel{(n+1)}}$$

$$a_{n+1}=\frac{(2n-1)}{(2n+3)}a_n$$

$a_1=\frac{(-1)a_0}{(3)}$ $\quad a_2=\frac{(1)a_1}{(5)}$ $\quad a_3=\frac{(3)a_2}{(7)}$ $\quad a_4=\frac{(5)a_3}{(9)}$

$a_1=\frac{-8}{3}$ $\quad a_2=\frac{1}{5}\left(\frac{-8}{3}\right)$ $\quad a_3=\left(\frac{\cancel{3}}{7}\right)\left(\frac{-8}{\cancel{15}_5}\right)$ $\quad a_4=\left(\frac{\cancel{5}}{9}\right)\left(\frac{-8}{\cancel{35}_7}\right)$

$a_2=\frac{-8}{15}$ $\quad a_3=\frac{-8}{35}$ $\quad a_4=\frac{-8}{63}$

$\therefore$ b) $y=x^{r_2}\left(8-\frac{8}{3}x-\frac{8}{15}x^2-\frac{8}{35}x^3-\frac{8}{63}x^4+\dots\right)$

closed form solution of a) $y=x^{-1/2}(7-7x)$

5. $y' = 2\sin y + 8e^x \quad , y(0) = 0$

$y' = \cancel{2\sin(0)} + 8\cancel{e^{(0)}} = 8$

$y'' = 2y'\cos y + 8e^x = 2(8)\cancel{\cos(0)} + 8\cancel{e^{x}} = 24$

$y''' = 2y''\cos y - 2(y')^2\sin y + 8e^x = 2(24)\cancel{\cos(0)} \cancel{-2(8)^2\sin(0)} + 8\cancel{e^{(0)}} = 56$

$\therefore \; y(x) = 8x + \frac{24}{2!}x^2 + \frac{56}{3!}x^3$

6. $(t^2 - 8t - 9)^2 x'' + (t^2 - 1)x' - tx = 0$

$(t-9)^2(t+1)^2 x'' + (t+1)(t-1)x' - tx = 0$

$x'' + \frac{\cancel{(t+1)}(t-1)}{(t-9)^2(t+1)^{\cancel{2}}}x' - \frac{t}{(t-9)^2(t+1)^2}x = 0$

$x'' + \frac{(t-1)}{(t-9)^2(t+1)}x' - \frac{t}{(t-9)^2(t+1)^2}x = 0$

since $(t+1)\left(\frac{(t-1)}{(t-9)^2(t+1)}\right) = \frac{t-1}{(t-9)^2}$ and $(t+1)^2\left(\frac{t}{(t-9)^2(t+1)^2}\right) = \frac{t}{(t-9)^2}$

are both defined at $t=-1$ $\quad \therefore t=-1$ is a regular singular point

since $(t-9)\left(\frac{(t-1)}{(t-9)^2(t+1)}\right) = \frac{t-1}{(t-9)(t+1)}$ is not defined at $t=9$, $t=9$ is an irregular singular point

C - All solutions remain bounded near t,

7. a) $F(s) = \frac{s+9}{s^2-9s+14}$

$a = \frac{9}{2}$

$b^2 = \sqrt{25/4}$

$b = 5/2$

$\frac{27}{2}c = \frac{5}{2}$

$c = \frac{5}{2}\left(\frac{2}{27}\right)$

$c = \frac{5}{27}$

$F(s) = \frac{s+9}{(s-9/2)^2-25/4}$

$F(s) = \frac{s+18/2+9/2-9/2}{(s-9/2)^2-25/4}$

$F(s) = \frac{s-9/2+27/2}{(s-9/2)^2-25/4}$

$* \; e^{at}\cosh(bt) = \frac{s-a}{(s-a)^2-b^2}$

$* \; e^{at}\sinh(bt) = \frac{b}{(s-a)^2-b^2}$

$F(s) = \frac{s-9/2}{(s-9/2)^2-25/4} + \frac{27/2}{(s-9/2)^2-25/4}$

$F(s) = \frac{s-9/2}{(s-9/2)^2-25/4} + \frac{\cancel{27/2}(5/\cancel{27})(27/5)}{(s-9/2)^2-25/4}$

$F(s) = \frac{s-9/2}{(s-9/2)^2-25/4} + \frac{27}{5}\,\frac{5/2}{(s-9/2)^2-25/4}$

$\boxed{f(t) = e^{9t/2}\cosh\left(\frac{5t}{2}\right) + \frac{27}{5}e^{9t/2}\sinh\left(\frac{5t}{2}\right)}$

b) $F(s) = \frac{2s+7}{s^2+s+9}$

$a = \frac{-1}{2}$

$b^2 = \frac{35}{4}$

$b = \frac{\sqrt{35}}{2}$

$6c = \frac{\sqrt{35}}{2}$

$c = \frac{\sqrt{35}}{12}$

$F(s) = \frac{2(s+1/2)+6}{(s+1/2)^2+35/4}$

$F(s) = \frac{2(s+1/2)}{(s+1/2)^2+35/4} + \frac{6}{(s+1/2)^2+35/4}$

$* \; e^{at}\sin(bt) = \frac{b}{(s-a)^2+b^2}$

$F(s) = 2\,\frac{s+1/2}{(s+1/2)^2+35/4} + \frac{12}{\sqrt{35}}\,\frac{\sqrt{35}/2}{(s+1/2)^2+35/4}$

$* \; e^{at}\cos(bt) = \frac{s-a}{(s-a)^2+b^2}$

$\boxed{f(t) = 2e^{-t/2}\cos\left(\frac{\sqrt{35}\,t}{2}\right) + \frac{12}{\sqrt{35}}e^{-t/2}\sin\left(\frac{\sqrt{35}\,t}{2}\right)}$

c) $F_{(s)} = \frac{s+9}{s^2-14s+49}$ $\quad$ $* \; t^n e^{at} = \frac{n!}{(s-a)^{n+1}}$

$F_{(s)} = \frac{s+9}{(s-7)^2}$

$F_{(s)} = \frac{s-7}{(s-7)^2} + \frac{16}{(s-7)^2}$

$F_{(s)} = \frac{1}{s-7} + \frac{16}{(s-7)^2}$

$\boxed{f_{(t)} = e^{7t} + 16te^{7t}}$

8. a) $f_{(t)} = 2e^{-4t} + 8t^2 + 7t + 7$ $\quad$ $* \; e^{at} = \frac{1}{s-a}$ $\quad$ $* \; t^n = \frac{n!}{s^{n+1}}$ $\quad$ $* \; n = \frac{1}{s}$

$F_{(s)} = \frac{2}{s+4} + \frac{8(2)}{s^3} + \frac{7}{s^2} + \frac{7}{s}$

$F_{(s)} = \frac{2}{s+4} + \frac{16}{s^3} + \frac{7}{s^2} + \frac{7}{s}$

b) $f_{(t)} = 2e^{4t}\sin(8t) + 7t^3 + 7e^t$ $\quad$ $* \; e^{at}\sin(bt) = \frac{b}{(s-a)^2+b^2}$ $\quad$ $* \; t^n = \frac{n!}{s^{n+1}}$ $\quad$ $* \; e^{at} = \frac{1}{s-a}$

$F_{(s)} = 2\left(\frac{8}{(s-4)^2+64}\right) + \frac{7(6)}{s^4} + \frac{7}{s-1}$

$F_{(s)} = \frac{16}{(s-4)^2+64} + \frac{42}{s^4} + \frac{7}{s-1}$

c) $f_{(t)} = 2te^{-4t}\sin(8t)$ $\quad$ $* \; t^n f_{(t)} = (-1)^n F^{(n)}_{(s)}$

where $f_{(t)} = 2e^{-4t}\sin(8t)$ and $n=1$

$\therefore F_{(s)} = \frac{-16}{(s+4)^2+64}$

$F_{(s)} = -16[(s+4)^2+64]^{-1}$

$F'_{(s)} = -16[(s+4)^2+64]^{-2}(2)(s+4)$

$F'_{(s)} = \frac{32(s+4)}{[(s+4)^2+64]^2}$

9. a) $y'' + 9y' + 3y$, $y(0) = 8$, $y'(0) = 7$

$\mathcal{L}(y''+9y'+3y)$

$= \mathcal{L}(y'') + 9\mathcal{L}(y') + 3\mathcal{L}(y)$

$= s^2Y - s\overset{8}{\cancel{y(0)}} - \overset{7}{\cancel{y'(0)}} + 9(sY - \overset{8}{\cancel{y(0)}}) + 3Y$

$= s^2Y - 8s - 7 + 9sY - 72 + 3Y$

$= s^2Y + 9sY + 3Y - 8s - 79$

b) $3y'' - y' + 4y$, $y(0) = -9$, $y'(0) = 8$

$\mathcal{L}(3y''-y'+4y)$

$= 3\mathcal{L}(y'') - \mathcal{L}(y') + 4\mathcal{L}(y)$

$= 3(s^2Y - s\overset{-9}{\cancel{y(0)}} - \overset{8}{\cancel{y'(0)}}) - (sY - \overset{-9}{\cancel{y(0)}}) + 4Y$

$= 3s^2Y + 27s - 24 - sY - 9 + 4Y$

$= 3s^2Y - sY + 4Y + 27s - 33$

c) $4y'' - 13y' - 7y$, $y(0) = -8$, $y'(0) = 9$

$\mathcal{L}(4y''-13y'-7y)$

$= 4\mathcal{L}(y'') - 13\mathcal{L}(y') - 7\mathcal{L}(y)$

$= 4(s^2Y - s\overset{-8}{\cancel{y(0)}} - \overset{9}{\cancel{y'(0)}}) - 13(sY - \overset{-8}{\cancel{y(0)}}) - 7Y$

$= 4s^2Y + 32s - 36 - 13sY - 104 - 7Y$

$= 4s^2Y - 13sY - 7Y + 32s - 140$