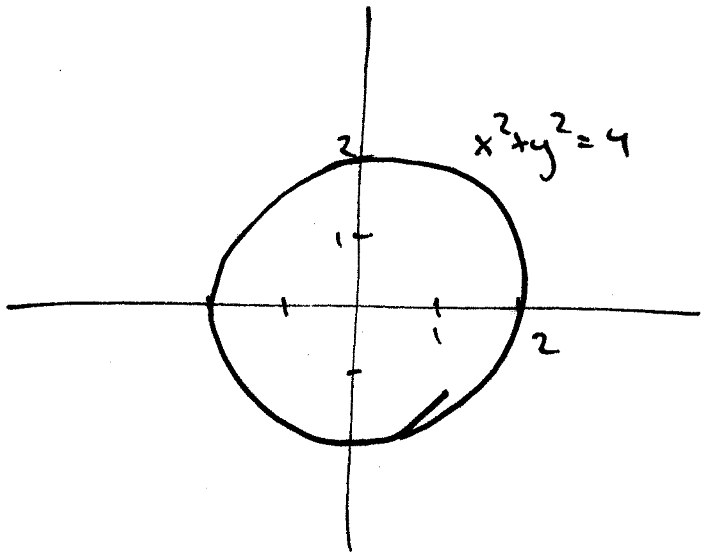
Lecture 10

Implicit differentiation

Consider $x^2 + y^2 = 4$. Plot all (x,y) satisfying this equation:



This is not a (graph of) a function; rather, it consists of two functions:

$y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$.

<u>Find</u> the slope at $(1, \sqrt{3})$.

<u>Sol^n 1</u> $y' = \frac{1}{2} \cdot \frac{1}{\sqrt{4-x^2}} \cdot (-2x)$ (Chain Rule)

$= \frac{-x}{\sqrt{4-x^2}}$

For $x=1$ get $y' = \frac{-1}{\sqrt{3}}$.

$\underline{\text{Sol}^n\ 2}$: Differentiate both sides of $x^2+y^2=4$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(4) \implies$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}(4) \implies$$

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies$$

$$2y\frac{dy}{dx} = -2x \implies$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

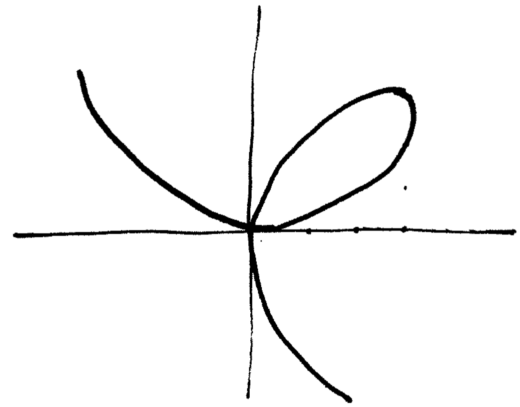
$$= -\frac{x}{y}$$

At $(1,\sqrt{3})$ get $\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$.

This is called <u>implicit</u> differentiation.

Sometimes we can't solve for y.

<u>Ex</u> $x^3 + y^3 = 6xy$



Find the slope at $(3,3)$.

<u>Sol^n</u>

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy) \implies$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6y + 6x\frac{dy}{dx} \implies$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2 \implies$$

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}.$$

At $(3,3)$ get $\frac{dy}{dx} = \frac{2\cdot 3 - 3^2}{3^2 - 2\cdot 3} = \frac{-3}{3} = -1.$

Find all horizontal tangent lines to
$x^3 + y^3 = 6xy$.

<u>Solⁿ</u>

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = 0 \implies$$

$$2y - x^2 = 0 \implies$$

$$x^2 = 2y \implies$$

$$y = \tfrac{1}{2}x^2.$$

Substitute into original curve:

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right) \implies$$

$$x^3 + \frac{x^6}{8} = 3x^3 \implies$$

$$x^3\left(1 + \frac{x^3}{8} - 3\right) = 0 \implies x = 0$$

or $x^3 = 16$

So $x = \sqrt[3]{16}$.

$y = \tfrac{1}{2}x^2 = \sqrt[3]{32}$

Answer: at $(0,0)$ and
$(\sqrt[3]{16}, \sqrt[3]{32})$.

Application: Logarithmic differentiation.

What is the derivative of $f(x) = \ln(x)$?

$y = \ln x$ means $(x > 0)$

$e^y = x$. Differentiate :

$$\frac{d}{dx} e^y = \frac{d}{dx} x \implies$$

$$e^y \cdot \frac{dy}{dx} = 1 \implies$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$

More generally: $\frac{d}{dx} \log_a(x) = \frac{1}{x \cdot \ln(a)}$

(recall: $\frac{d}{dx} a^x = \ln(a) \cdot a^x$)

However, for $y = \ln(-x)$, $x < 0$.

we also get $\frac{dy}{dx} = \frac{-1}{-x}$, so:

$$= \frac{1}{x}$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \qquad x \neq 0.$$

Power Rule, revisited

Claim: $$\frac{d}{dx} x^n = n x^{n-1}$$

all $n \in \mathbb{R}$.

Proof $$y = x^n \overset{x \neq 0}{\Longrightarrow}$$

$$\ln|y| = \ln|x^n|$$
$$= n \ln|x| \qquad \text{So}$$

$$\frac{d}{dx} \ln|y| = \frac{d}{dx} n \ln|x| \Rightarrow$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = n \cdot \frac{1}{x} \Rightarrow$$

$$\frac{dy}{dx} = n \cdot \frac{y}{x}$$
$$= n \cdot \frac{x^n}{x}$$
$$= n \cdot x^{n-1}.$$

Application: derivatives of inverse trig functions.

Find the derivative of $\arcsin(x)$.

Sol^n Recall: $y = \arcsin(x) \iff$

$\sin(y) = x$. (on $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$)

Differentiate both sides:

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}x \implies$$

$$\cos(y)\cdot\frac{dy}{dx} = 1 \qquad \text{so}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Since $\cos^2(y) + \sin^2(y) = 1$,

$$\cos(y) = \sqrt{1-\sin^2(y)}$$

$$= \sqrt{1-x^2}$$

So $$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}.$$

(Exercise: do arccos and arctan.)