Lecture 9

Chain Rule

Consider $y = f(g(x))$. What is dy/dx ?

y is a composite function : write

$y = f(u)$ $u = g(x)$.

Using the notation $y'(u) = \dfrac{dy}{du}$ $u'(x) = \dfrac{du}{dx}$

get $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$:

derivative of $f(g(x))$ is the product of the derivatives $f'(u)$ and $g'(x)$:

$$\boxed{\frac{d}{dx} f(g(x)) = f'(u)\, g'(x) = f'(g(x)) \cdot g'(x).}$$

Ex Find the derivative of $h(x) = \sqrt[3]{2x^2 - 3x}$.

Sol^n Write $h(u) = \sqrt[3]{u}$, $u(x) = 2x^2 - 3x$.

Then $h'(u) = \frac{1}{3} u^{-2/3} = \dfrac{1}{3\sqrt[3]{u^2}}$ $u'(x) = 4x - 3$.

So $h'(x) = h'(2x^2 - 3x) \cdot u'(x)$

$$= \frac{1}{3\sqrt[3]{(2x^2 - 3x)^2}} \cdot (4x - 3)$$

Population Growth

Population of bacteria at time t: $P(t)$.

Exponential growth means: $P'(t) = k \cdot P(t)$

(growth rate is proportional to population size).

To find $P(t)$, we recall that if

$f(x) = e^{ax}$ (for $a \in \mathbb{R}$) then

$f'(x) = a \cdot e^{ax}$ (by the chain rule), so that

indeed $f'(x) = a \cdot f(x)$.

More generally, if $f(x) = C \cdot e^{ax}$ $\quad C, a \in \mathbb{R}$

then $f'(x) = aC \cdot e^{ax} = a f(x)$.

Thus exponential growth functions have the form $P(t) = C \cdot e^{kt}$ for $C, k \in \mathbb{R}$

Note: $P(0) = C \cdot e^{0} = C$ so C is the size of the initial population.

k is called the proportionality constant

Ex Find the derivative of $\tan(\cos x)$.

Sol^n $h(u) = \tan(u)$ $u(x) = \cos(x)$

$h'(u) = \sec^2(u)$ $u'(x) = -\sin(x)$

So $h'(x) = \sec^2(\cos(x)) \cdot -\sin(x)$

$\left(= \dfrac{-\sin(x)}{\cos^2(\cos(x))} \right)$.

Ex Find the derivative of $\sqrt{e^x}$.

$h(u) = \sqrt{u}$ $u(x) = e^x$

$h'(u) = \dfrac{1}{2\sqrt{u}}$ $u'(x) = e^x$

$$h'(x) = \frac{1}{2\sqrt{e^x}} \cdot e^x = \frac{\sqrt{e^x}}{2}.$$

(Alternatively: $\sqrt{e^x} = (e^x)^{\frac{1}{2}} = e^{\frac{1}{2}x}$.

Now $h(u) = e^u$, $u(x) = \frac{1}{2}x$

$h'(u) = e^u$, $u'(x) = \frac{1}{2}$

So $h'(x) = e^{\frac{1}{2}x} \cdot \frac{1}{2}$.)

Exponential Decay

Half life of radium-226 is 1590 years.

(a) A sample has a mass of 100 mg. How much radium remains after t years?

Sol^n $\quad m(t) = Ce^{kt}$. $\qquad C = m(0) = 100$.

$$= 100e^{kt}.$$

$$100 \cdot e^{k(t+1590)} = 1/2 \cdot 100 e^{kt} \quad \text{so}$$

$$e^{k \cdot 1590} = 1/2 \quad \text{so}$$

$$k \cdot 1590 = \ln 1/2 \quad \text{and}$$

$$k = \ln 1/2 / 1590$$

$$= -\frac{\ln 2}{1590}.$$

Now $\quad m(t) = 100 \cdot e^{-\frac{\ln 2}{1590} t}$

$$= 100 \cdot 2^{-\frac{t}{1590}}$$

$$\left(= 100 \cdot \left(\tfrac{1}{2}\right)^{t/1590} \right).$$