$\boxed{\text{Lecture 8}}$ More Rules.

Product Rule:

$$\boxed{\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)}$$

Proof:

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} =$$

$$\lim_{h \to 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right) =$$

$$f(x)g'(x) + g(x)f'(x).$$

Example

$$\frac{d}{dx} \underbrace{(x^2+4x)}_{f(x)} \underbrace{\sqrt{x}}_{g(x)} =$$

$$\underbrace{(x^2+4x)}_{f(x)} \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{g'(x)} + \underbrace{(2x+4)}_{f'(x)} \cdot \underbrace{\sqrt{x}}_{g(x)}$$

(Note: could also use $(x^2+4x)\sqrt{x} = \sqrt{x}\,x^2 + 4x\sqrt{x}$

$= x^{2\frac{1}{2}} + 4x^{1/2}$.)

Quotient Rule:

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

(Exercise: prove this!)

Ex Find $\frac{d}{dx}\frac{1}{x}$.

Sol^n 1 : Power Rule:

$\frac{1}{x} = x^{-1}$ so

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -1 \cdot x^{-2} = -\frac{1}{x^2}.$$

Sol^n 2 : Quotient Rule

$\frac{1}{x} = \frac{f(x)}{g(x)}$ with $f(x)=1$, $g(x)=x$
$f'(x)=0$, $g'(x)=1$

so $$\frac{d}{dx}\frac{1}{x} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$= \frac{1 \cdot 0 - 1 \cdot 1}{x^2}$$

$$= -\frac{1}{x^2}.$$

Derivative of sin(x)

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} =$$

$$\lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$\lim_{h\to 0} \left(\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \right) =$$

$$\lim_{h\to 0} \left(\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right) . \qquad (*)$$

So we need to know $\lim_{h\to 0} \frac{\cos(h) - 1}{h}$ and $\lim_{h\to 0} \frac{\sin(h)}{h}$.

Fact (show in class, don't need to memorize this for now): $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$,

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0 .$$

Thus $(*) = \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x) .$$

So $\frac{d}{dx} \sin(x) = \cos(x)$.

Similarly we can show (exercise!) that $\frac{d}{dx}\cos(x) = -\sin(x)$

(Note the $-$ sign!)

Q. What is $\frac{d}{dx}\tan(x)$?

A. Use the quotient rule:

$\tan(x) = \frac{\sin(x)}{\cos(x)}$. So

$$\frac{d}{dx}\tan(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2(x)}$$

$\left(= \sec^2(x)\right)$.

Can find derivatives of the three other trig functions in the same way.