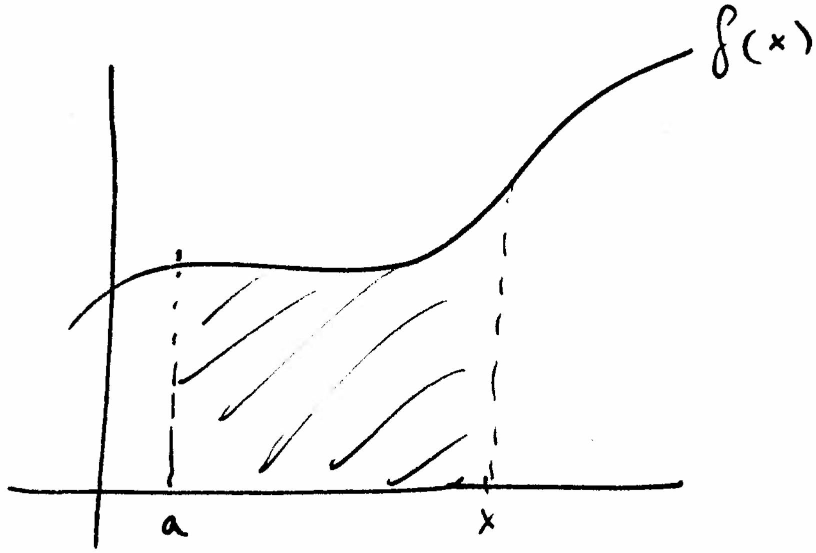
Lecture 6 — FToC Part 1

① Area-so-far



$F(x)$ = area under f between a and x
(here a is fixed, x varies).

So: $$F(x) = \int_a^x f(t)\,dt.$$ (Also write $F_a(x)$.)

Ex $f(x) = x^2$, $a = 1$

$$F_a(x) = \int_1^x f(t)\,dt = \int_1^x t^2\,dt = \frac{1}{3}t^3 \Big|_1^x$$

$$= \frac{1}{3}x^3 - \frac{1}{3}$$

E.g. $F(2) = \frac{1}{3}\cdot 2^3 - \frac{1}{3} = \frac{7}{3}$

$F(0) = \frac{1}{3}\cdot 0^3 - \frac{1}{3} = -\frac{1}{3}$.

F.T.C Part 1

Let f be integrable on $[a,b]$. Then for $x \in [a,b]$ we have

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

In other words: $F_a'(x) = f(x)$, so F_a is an anti-derivative of f.

Ex $f(x) = x^2$ $\quad F_a(x) = \frac{1}{3}x^3 - \frac{1}{3}a^3$

Then $F_a'(x) = x^2$.

(Note: the term $\frac{1}{3}a^3$ is constant so vanishes when we differentiate.)

Ex If $f(t) = |v(t)|$ is the velocity at time t, then $F_0(x) = \int_0^x f(t)\,dt$ represents the distance travelled up to time x.

FToC states that if a function f is integrable, then it has an anti-derivative

$$F_a(x) = \int_a^x f(t)\,dt \; .$$

However, this is true for any a, so a given f has many anti-derivatives.

To remove the reference to a, we write

$$\int f(t)\,dt$$

for the <u>general anti-derivative</u> of f (if it exists).

If $F(x)$ is any anti-derivative of f, we have $F(x) = \int f(x)\,dx + c$ for an arbitrary constant c.

<u>Ex</u> $\quad \int 2x\,dx = x^2 + C \; .$

② More about derivatives.

Standard Problem 1. Find the equation of the tangent line to $f(x) = x^2 + 3$ at $x = 2$.

Sol^n $f'(x) = 2x$. So the slope is:

$$a = f'(2) = 2 \cdot 2 = 4.$$

Given point: $(2, f(2)) = (2, 7)$

The form is $y = 4x + b$; plug in $x = 2,\ y = 7$:

$7 = 4 \cdot 2 + b \rightarrow b = 1$. So the equation is $y = 4x - 1$.

Standard Problem 2. Using the definition, find the derivative of $f(x) = \sqrt{x}$.

Sol^n $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h \cdot (\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Note the domain of f is $[0, \infty)$,
〃 〃 〃 f' 〃 $(0, \infty)$.

This is a special case of the <u>Power Rule</u>:

<u>Thm</u> Let $n \in \mathbb{R}$. Then

$$\frac{d}{dx} x^n = n \cdot x^{n-1}.$$

(Our case was $n = 1/2$ so $x^{1/2} = \sqrt{x}$.)

Using the FToC, we get:

<u>Thm</u> Let $n \in \mathbb{R}$, $n \neq -1$. Then

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}$$

Note the exception $n = -1$. Then we'd get $\frac{1}{-1+1} x^0$, obvious nonsense.

So we don't know how to integrate $\frac{1}{x}$ yet.