Lecture 5 — Fundamental Theorem, and Applications

① So far, we defined:

- derivative of f at a (slope)
- definite integral (area).

We now study the connections between these concepts, and look at examples / applications.

Recall: given f, and $a \in \mathrm{dom}(f)$, we let

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{(if the limit exists)}.$$

Suppose now that $f'(a)$ exists for all $a \in B$ for some subset $B \subseteq \mathrm{dom}(f)$.

Set $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

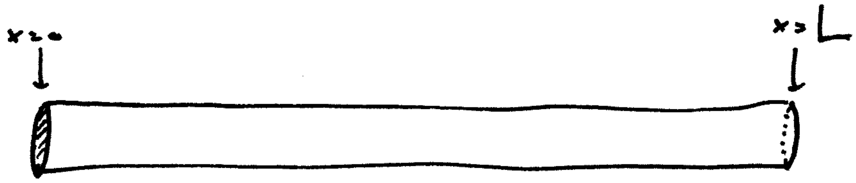
This new function is the derivative of f.

* Alternative notation: $\dfrac{df}{dx}$.

* $f'(x)$ is the function describing the rate of change of f.

Example A. A car is driving on a straight road. If $s(t)$ describes the position at time t, then $s'(t)$ describes the (instantaneous) velocity at time t.

Example B.



Solid metal rod

If $m(x)$ describes the mass of the rod up to length x, then $m'(x)$ describes the (linear) density (in g/m) of the rod at x.

Example C. If $P(t)$ describes the number of bacteria in a petri dish at time t, then $P'(t)$ describes the population growth at time t.

② Towards the F.T.C.

Consider $f(x) = 3x$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h) - 3x}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3.$$

(So: the slope of f is <u>constant</u>.)

Now compute $\int_a^b f'(x)\,dx$:

$$\int_a^b f'(x)\,dx = \int_a^b 3\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} 3 \cdot \frac{b-a}{n}$$

$$= \lim_{n \to \infty} n \cdot 3 \cdot \frac{b-a}{n} = \lim_{n \to \infty} 3(b-a) = 3b - 3a$$

$$= f(b) - f(a).$$

So: $\int_a^b f'(x)\,dx = f(b) - f(a)$

This holds in fact for <u>every</u> differentiable function f.

Fundamental Theorem of Calculus (Part 2)

Suppose f is differentiable on $[a,b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Or: Suppose f is continuous on $[a,b]$ and suppose that F is a function with $F' = f$.

Then $$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is first of all a profound connection between slope and area / differentiating and integrating: the definite integral "undoes" what differentiating does.

Or: the area under the graph of $f'(x)$ is simply the number $f(b) - f(a)$.

Second: it is a method / shortcut for computing areas, avoiding Riemann Sums.

Example (Rod Revisited)

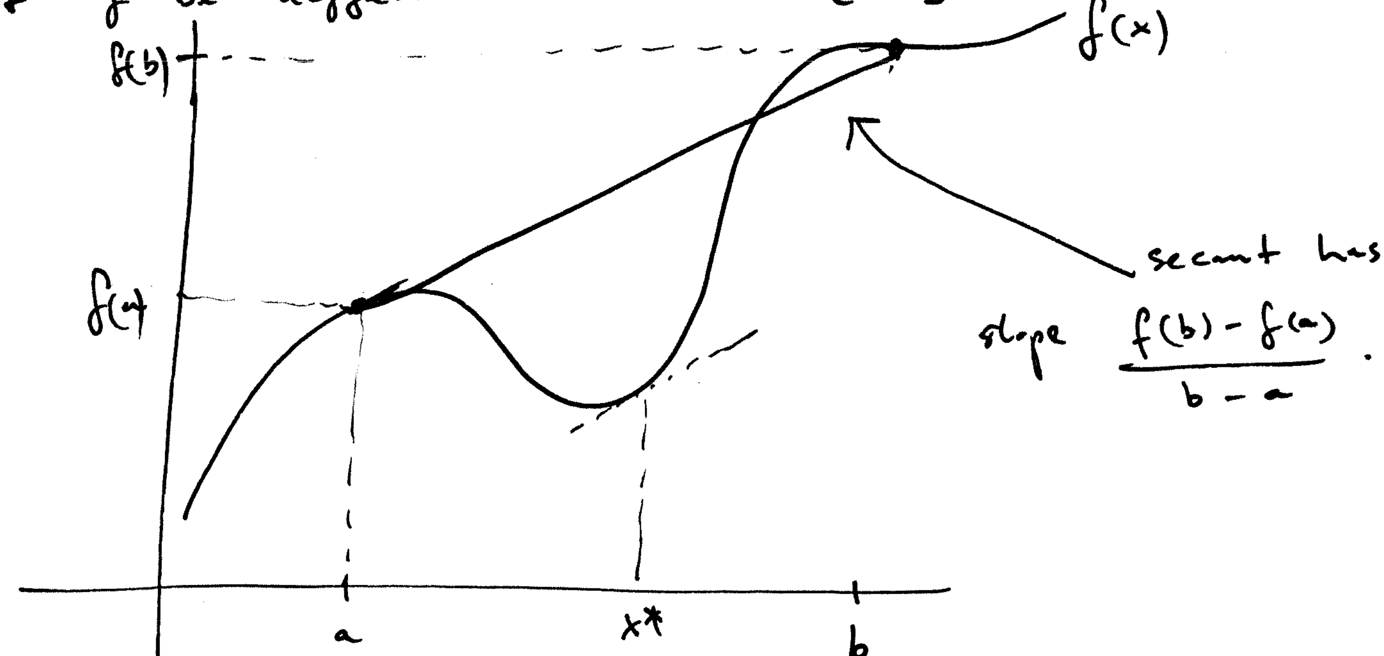
Let $m'(x)$ be the density of a rod at x.
To find the total mass of the rod,
we compute $\displaystyle\int_0^L m'(x)\,dx = m(L)$.

More generally, the mass of the segment
from $x=a$ to $x=b$ is

$$\int_a^b m'(x)\,dx = m(b) - m(a).$$

Application: <u>Mean Value Theorem.</u>

Let f be differentiable on $[a,b]$.



Then there is an $x^* \in [a,b]$ with $f'(x^*) = \dfrac{f(b)-f(a)}{b-a}$