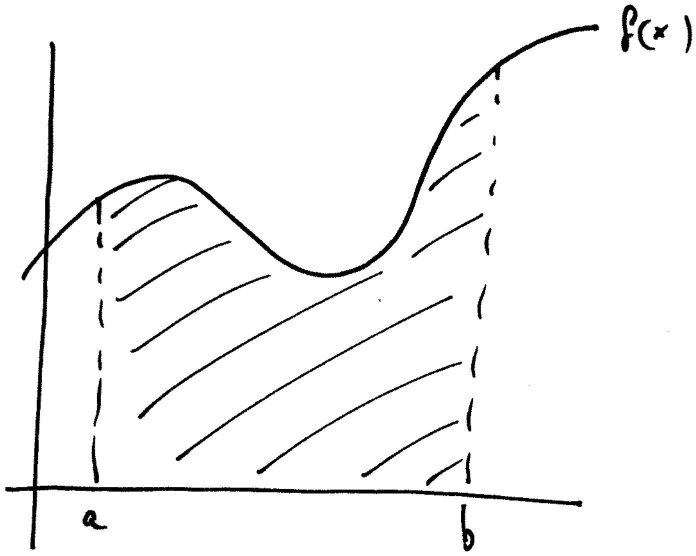
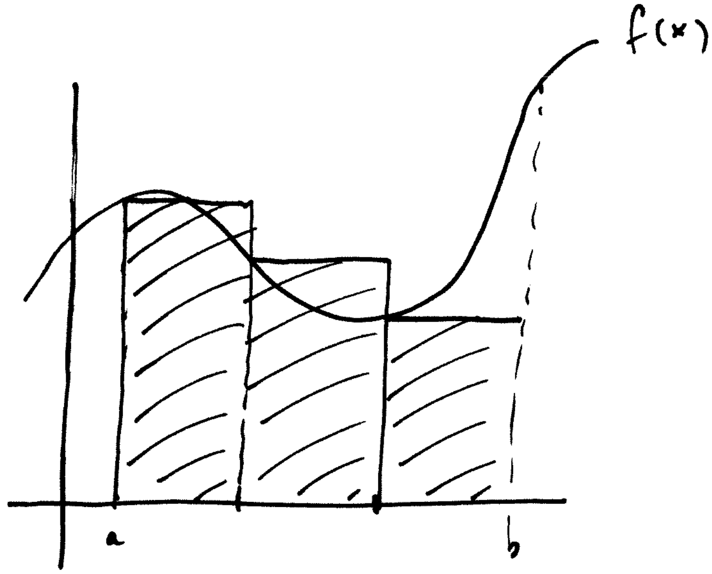
Lecture 4 Riemann Sums

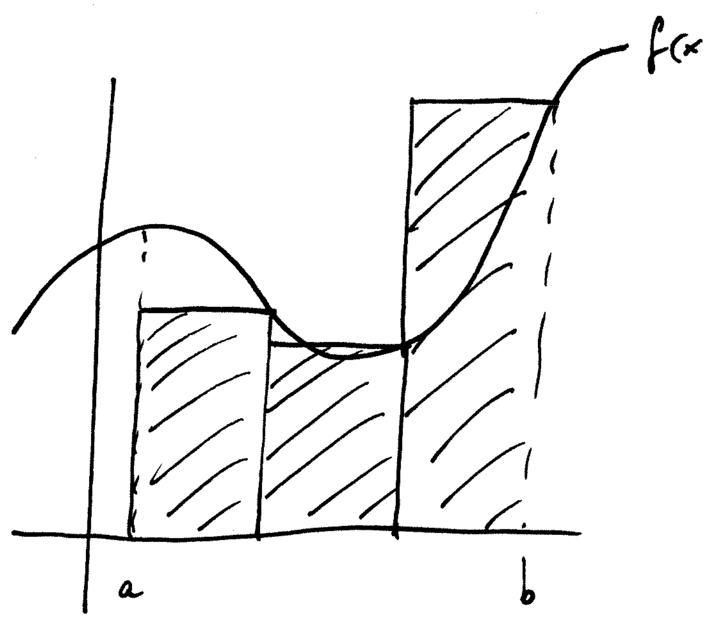


Problem: find the area under the graph of f between x = a and x = b.

Approach: approximate the area by rectangles.
(For illustration, I use 3 rectangles in the picture below.)



Left Riemann sum
L_3



Right Riemann sum
R_3

Compute the area of these rectangles.

<u>Width</u> of one rectangle: $\Delta x = \dfrac{b-a}{n}$

(in our example, $n = 3$).

<u>Height</u> of rectangle 1: $f(a)$

" " " 2: $f(a+\Delta x)$

" " " 3: $f(a+2x)$

<u>Area</u> of rectangle 1: $f(a) \cdot \Delta x$

" " " 2: $f(a+\Delta x) \cdot \Delta x$

" " " 3: $f(a+2\Delta x) \cdot \Delta x$

- $L_3 = f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x$.

Similarly, we can use the values $f(a+\Delta x)$, $f(a+2\Delta x)$, $f(b)$ for the heights, to get

- $R_3 = f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + f(b)\Delta x$.

Now as n gets larger, the area of the rectangles better approximates the actual area.

Letting L_n be the area of the rectangles

$$L_n = f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x$$

Similarly:

$$R_n = f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x + f(b)\Delta x.$$

Again, $\Delta x = \frac{b-a}{n}$.

<u>Now</u> define:

$$\int_a^b f(x)\,dx =_{def} \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n .$$

area, def. integral

<u>Note</u>: this is a limit, so it may or may not exist. If it does exist, we say that f is <u>integrable</u> on $[a,b]$.

Example : we compute $\int_0^1 x^2 dx$. Note: $\Delta x = \frac{1}{n}$.

$$L_n = f(0)\Delta x + f(\Delta x)\Delta x + f(2\Delta x)\Delta x + \cdots + f(1-\Delta x)\Delta x$$
$$= 0^2 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n}$$
$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right)$$

Fact : $1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)\,n\,(2n-1)}{6}$.

This gives

$$\frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) = \frac{(n-1)\,n\,(2n-1)}{6n^3}$$

$$= \frac{(n-1)(2n-1)}{6n^2} = \frac{2n^2 - 3n + 1}{6n^2} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

So

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}.$$

Exercise : compute $\lim_{n\to\infty} R_n$.

Properties of the definite integral

(i) $$\int_a^a f(x)\,dx = 0$$

(ii) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(iii) $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(iv) $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

(v) $$\int_a^b c\,dx = c(b-a).$$