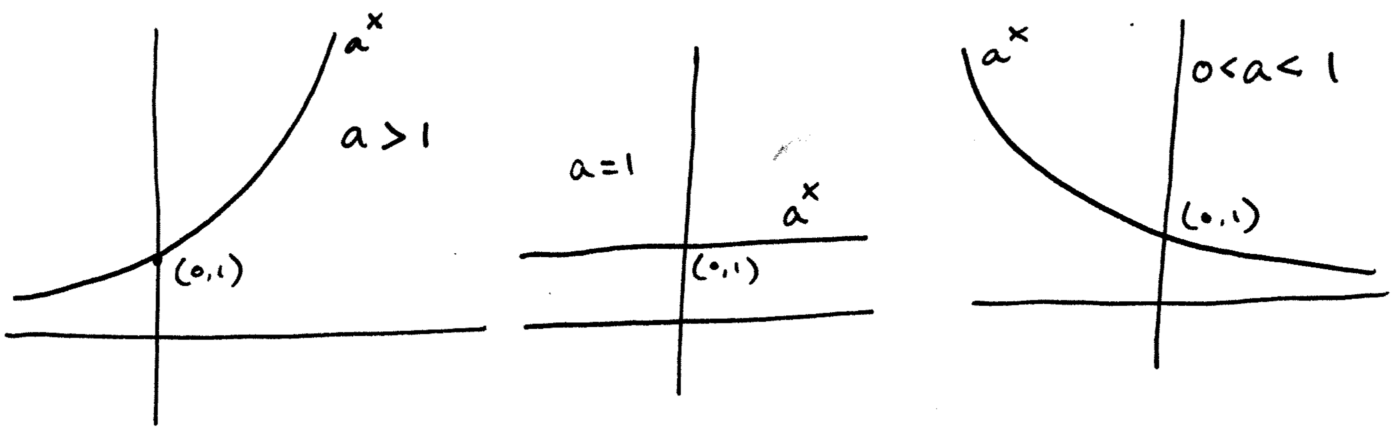
Lecture 3

1 Exponential functions and Logarithms

Graph of a^x :



Laws of exponents :

(i) $a^0 = 1$

(ii) $a^{b+c} = a^b \cdot a^c$

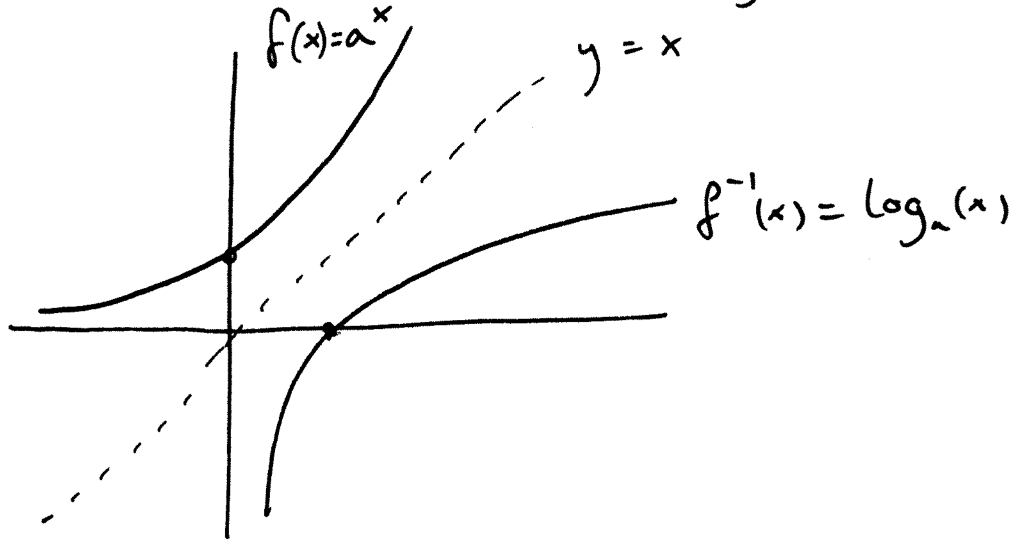
(iii) $a^{b-c} = a^b / a^c$

(iv) $(a^b)^c = a^{bc}$

Note : as a gets larger, the slope of the tangent line of a^x at $(0,1)$ becomes larger as well.

Def^n e is the number for which e^x has slope 1 at $(0,1)$.

Since a^x is 1-1 it has an inverse on $(0,\infty)$:
we call this function $\log_a(x)$.



By definition we have:

$$a^{\log_a(x)} = x \qquad x \in (0,\infty)$$
$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

<u>Laws of Logarithms</u>:

(i) $\log_a(1) = 0$
(ii) $\log_a(bc) = \log_a(b) + \log_a(c)$
(iii) $\log_a(b/c) = \log_a(b) - \log_a(c)$
(iv) $\log_a(b^c) = c \cdot \log_a(b)$

<u>Notation</u>:

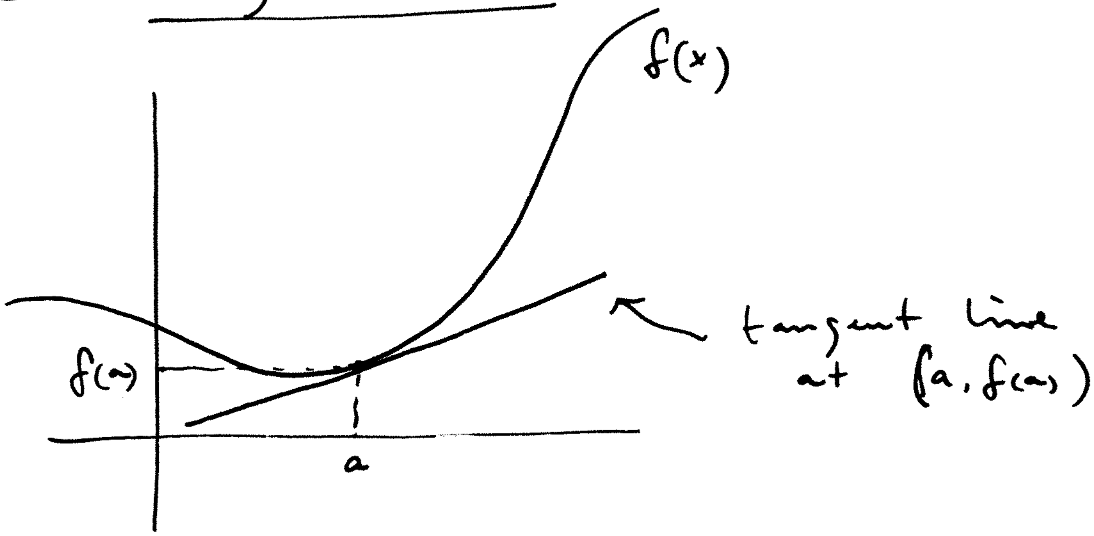
$\log_e(x) = \underline{\ln} x$

<u>N</u>atural <u>l</u>ogarithm.

The following is often useful ("change of base"):

$$a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a)\cdot x}$$

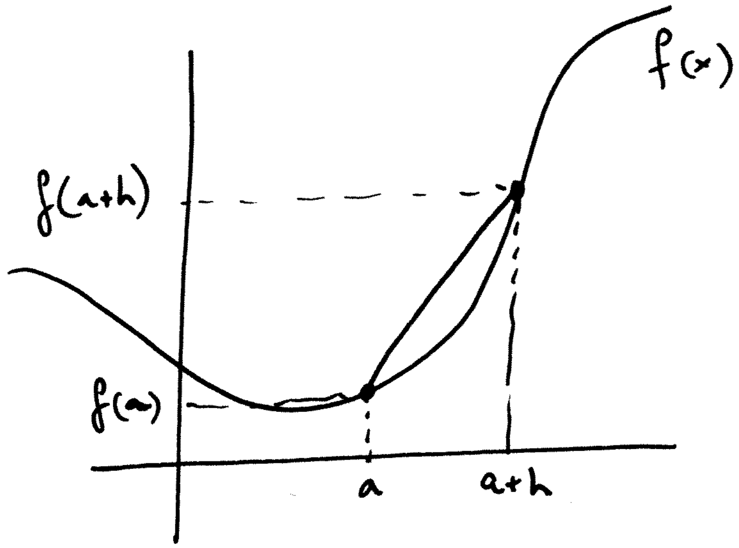
② Tangent Lines



Intuition: the tangent line at (a, f(a)) is the best linear approximation to f at x = a. Its slope is the slope of f at x = a, representing the instantaneous rate of change of f at a.

Not always does this make sense: for example the function $f(x) = |x|$ has no well-defined tangent line at $x = 0$.

Saying directly what the slope (of the tangent line) of f at x = a is is difficult; we therefore approximate it using secant lines.



$\boxed{\text{a secant line for } f \text{ at } x = a.}$

Calculating the slope of the secant line is easy:

$$\text{slope} = \frac{f(a+h) - f(a)}{h} \qquad \text{(rise over run)}.$$

Now as h gets smaller, the slope of the secant approximates the slope of the actual tangent line:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'(a)$ is called the <u>derivative</u> of f at $x=a$.

Example: Find $f'(3)$ for $f(x) = x^2 + 3x - 5$.

Sol^n $f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

$$= \lim_{h \to 0} \frac{[(3+h)^2 + 3(3+h) - 5] - [3^2 + 3 \cdot 3 - 5]}{h}$$

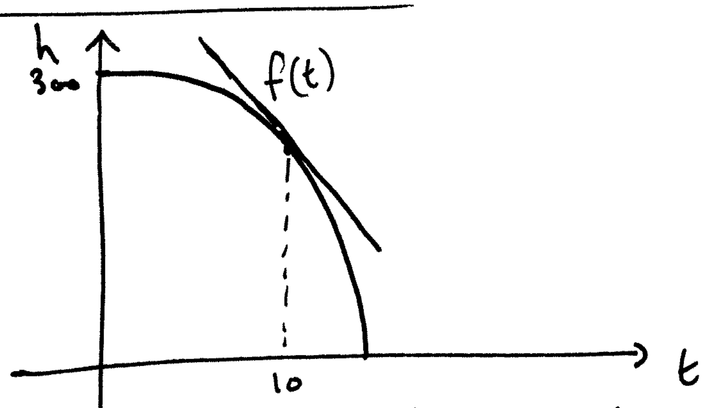
$$= \lim_{h \to 0} \frac{(h^2 + 6h + 9) + (9 + 3h) - 5 - (9 + 9 - 5)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 9h}{h}$$

$$= \lim_{h \to 0} h + 9$$

$$= 9$$

Application



An object is falling from a height of 300m. At $t = 10$ (after 10 seconds) the slope of the tangent line represents the velocity of the object.