$\boxed{\text{Lecture 2}}$ More functions, and more details

① New functions from old.

(i) Piecewise functions

e.g. $$f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1/x & \text{if } x < 0 \end{cases}$$

or $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

(ii) Sums Given f, g (with same domain and codomain) define

$$h(x) = f(x) + g(x) \quad (\text{Sum of } f \text{ and } g).$$

Ex $f(x) + 7$ (shift graph up by 7 units)

(iii) Products: $h(x) = f(x) \cdot g(x)$

Ex · $2 \cdot f(x)$ (vertical rescaling by factor 2)
· $-f(x)$ (reflect in x-axis)

(iv) Quotients $h(x) = \dfrac{f(x)}{g(x)}$ (different domain!)

Ex $\csc(x) = \dfrac{1}{\sin x}$

(v) Composition $(f \circ g)(x) = f(g(x))$

Ex $f(x) = 3^x$ $g(x) = x - 4$

$f \circ g$ is $(f \circ g)(x) = 3^{x-4}$

(shift 4 units to the right)

Ex $f(x) = \sin(x)$ $g(t) = 2\pi t$

$(f \circ g)(t) = \sin(2\pi t)$

(vi) Inverse. Idea: given f, find a function g which "undoes" f, in the sense that

$g(f(x)) = x$

$f(g(x)) = x$

Equiv: $y = f(x) \iff x = g(y)$

Notation: $g = f^{-1}$.

Ex $f(x) = \dfrac{x+3}{x-4}$. Find f^{-1} (if it exists)

Sol'n $y = \dfrac{x+3}{x-4}$ $x \neq 4$

$\Rightarrow (x-4)y = x+3$

$\Rightarrow xy - 4y = x+3$

$\Rightarrow xy - x = 3 + 4y$

$\Rightarrow x(y-1) = 3 + 4y$

$\Rightarrow x = \dfrac{3+4y}{y-1}$ $y \neq 1$.

Ex $f(x) = e^x$. Then $f^{-1}(x) = \ln(x)$

(2) Properties of functions

Let $f: A \to B$, $A, B \subseteq \mathbb{R}$

(i) f is continuous if the graph of f does not have any "jumps". We make a rigorous definition later.

(ii) f is differentiable if for each $x \in A$, the graph of f has a well-defined slope at x.

(iii) f is increasing / decreasing if for all $x < y$ we have $f(x) < f(y)$ / $f(x) < f(y)$

(iv) f is even / odd if $f(x) = f(-x)$ / $f(x) = -f(-x)$

(v) f is periodic if there is a number P with $f(x+P) = f(x)$ for all $x \in A$.

(vii) f is one-to-one (also: injective) if

$$x \neq y \implies f(x) \neq f(y) \quad \text{for all } x, y \in A$$

In other words: f is injective if different inputs always result in different outputs.

Ex $f(x) = x^3 - 3$ is injective, but $g(x) = x^2 - 3$ is not.