

NOTE: - Only answers are provided here (and some proofs). On the test you **must** provide full and complete solutions to receive full marks.

1. Short Answer Problems

a) State the Principal Axis Theorem.

Solution: Let A be an $n \times n$ matrix. A is symmetric if and only if A is orthogonally diagonalizable.

b) Determine if $\langle p, q \rangle = p(-1)q(-1) + p(1)q(1)$ is an inner product for P_2 .

Solution: It is not an inner product since $\langle x^2 - 1, x^2 - 1 \rangle = 0$.

c) State the Rank-Nullity Theorem.

Solution: Suppose that \mathbb{V} is an n -dimensional vector space and that $L : \mathbb{V} \rightarrow \mathbb{W}$ is a linear mapping into a vector space \mathbb{W} . Then $\text{rank}(L) + \text{nullity}(L) = \dim \mathbb{V} = n$

d) Let A be a matrix such that A^T is row equivalent to

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a basis for $(\text{Col } A)^\perp$.

Solution: $(\text{Col } A)^\perp = \text{Null}(A^T)$. A basis for $\text{Null}(A^T)$ is $\left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. Let $L : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ be given by $L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^T$. Find the matrix for L with respect to the basis \mathcal{B} of $M_{2 \times 2}(\mathbb{R})$, where

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution: $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find an orthogonal matrix P that diagonalizes A and the corresponding diagonal matrix.

Solution: $P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

4. Find an \vec{x} that minimizes $\|A\vec{x} - \vec{b}\|$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution: $\vec{x} = \begin{bmatrix} 3 \\ -0.9 \end{bmatrix}$.

5. Let T be a linear operator on an inner product space \mathbb{V} , and suppose that $\langle \vec{x}, \vec{y} \rangle = \langle T(\vec{x}), T(\vec{y}) \rangle$ for all \vec{x} and \vec{y} in \mathbb{V} . Prove that T is an isomorphism.

Solution: We are given that T is linear. Let $\vec{x} \in \text{Null}(T)$. Then,

$$\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle \vec{0}, \vec{0} \rangle = 0$$

Hence $\vec{x} = \vec{0}$. Therefore, $\ker(T) = \{\vec{0}\}$ and so T is one-to-one. Since T is one-to-one and $T : \mathbb{V} \rightarrow \mathbb{V}$, by Theorem 2.4.4 we get that T is also onto. Hence, T is an isomorphism.

6. Let Q be an $n \times n$ orthogonal matrix, and let \vec{x} and \vec{y} be orthogonal vectors in \mathbb{R}^n . Show that $Q\vec{x}$ and $Q\vec{y}$ are orthogonal.

Solution: We have $(Q\vec{x}) \cdot (Q\vec{y}) = (Q\vec{x})^T(Q\vec{y}) = \vec{x}^T Q^T Q \vec{y} = \vec{x}^T \vec{y} = 0$

7. Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ be real vectors spaces and let $L : \mathbb{U} \rightarrow \mathbb{V}$ and $M : \mathbb{V} \rightarrow \mathbb{W}$ be linear mappings. Prove that if L and M are onto, then $M \circ L$ is onto.

Solution: Let $\vec{w} \in \mathbb{W}$. Since M is onto, there exists $\vec{v} \in \mathbb{V}$ such that $M(\vec{v}) = \vec{w}$. Similarly, since L is onto, there exists $\vec{u} \in \mathbb{U}$ such that $L(\vec{u}) = \vec{v}$. Thus, we have

$$(M \circ L)(\vec{u}) = M(L(\vec{u})) = M(\vec{v}) = \vec{w}$$

as required.

8. Consider P_2 with inner product $\langle p(x), q(x) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

a) Find the value of $\langle 1 - x - x^2, 1 + x^2 \rangle$.

Solution: $\langle 1 - x - x^2, 1 + x^2 \rangle = (1)(2) + (1)(1) + (-1)(2) = 1$.

b) Find the distance between $1 - x - x^2$ and $1 + x^2$.

Solution: The distance is $\|(1 - x - x^2) - (1 + x^2)\| = \sqrt{(-1)^2 + (0)^2 + (-3)^2} = \sqrt{10}$.

c) Determine the coordinates of $1 - 2x + x^2$ with respect to the orthonormal basis

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{6}}(2 - 3x^2) \right\}.$$

Solution: $[1 - 2x + x^2]_{\mathcal{B}} = \begin{bmatrix} 5/\sqrt{3} \\ -4/\sqrt{2} \\ -2/\sqrt{6} \end{bmatrix}$.

d) Given that $S = \{1 - x^2, \frac{1}{2}(x - x^2)\}$ is orthonormal, extend S to find an orthonormal basis for P_2 .

Solution: $\{1 - x^2, \frac{1}{2}(x - x^2), \frac{1}{2}(x + x^2)\}$ is an orthonormal basis of P_2 .

9. Let \mathbb{V} be a real inner product space with inner product $\langle \cdot, \cdot \rangle$ and let $\vec{u}, \vec{v} \in \mathbb{V}$.

Prove that $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ if and only if $\langle \vec{u}, \vec{v} \rangle = 0$.

Solution: We have

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle = \langle \vec{u}, \vec{u} + \vec{v} \rangle + \langle \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + 0 + 0 + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$

If $\vec{u}, \vec{v} \in \mathbb{R}^n$, then we have

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \|\vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2.$$

Hence $2\langle \vec{u}, \vec{v} \rangle = 0$ and $\langle \vec{u}, \vec{v} \rangle = 0$ as required.

10. Determine whether each of the following statements is true or false. Justify your choices with reasoning based on theorems, definitions, or counter-examples. Answers with no explanations will receive no marks.

a) There exists a linear operator $L : P_1 \rightarrow P_1$ such that $\text{Range}(L) = \ker(L)$.

Solution: TRUE

b) If A and B are $n \times n$ symmetric matrices, then AB is orthogonally diagonalizable

Solution: FALSE

c) Let A and B be $n \times n$ symmetric matrices. If there exists an orthogonal matrix P such that $P^T A P$ and $P^T B P$ are diagonal, then $AB = BA$.

Solution: TRUE

d) If $L : \mathbb{V} \rightarrow \mathbb{W}$ is an onto linear mapping and $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent in \mathbb{V} , then $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is linearly independent in \mathbb{W} .

Solution: FALSE

11. Let $T : \mathbb{V} \rightarrow \mathbb{V}$ be a linear mapping and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for \mathbb{V} such that $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is a basis for $\ker(T)$. Prove that there exists an isomorphism $S : \mathbb{V} \rightarrow \mathbb{V}$ such that $T \circ S \circ T = T$.

(You do not need to prove that your mapping S is linear).

Solution: By the proof of the Rank-Nullity Theorem, we know that $\{T(\vec{v}_1), \dots, T(\vec{v}_r)\}$ is a basis for $\text{Range}(T)$. Extend this to a basis $\{T(\vec{v}_1), \dots, T(\vec{v}_r), \vec{w}_{r+1}, \dots, \vec{w}_n\}$ for \mathbb{V} .

Define $S : \mathbb{V} \rightarrow \mathbb{V}$ by

$$S(c_1 T(\vec{v}_1) + \dots + c_r T(\vec{v}_r) + c_{r+1} \vec{w}_{r+1} + \dots + c_n \vec{w}_n) = c_1 \vec{v}_1 + \dots + c_r \vec{v}_r + c_{r+1} \vec{v}_{r+1} + \dots + c_n \vec{v}_n$$

S is clearly linear and, if $\vec{y} \in \ker(S)$, then

$$\vec{0} = S(y_1 T(\vec{v}_1) + \dots + y_r T(\vec{v}_r) + y_{r+1} \vec{w}_{r+1} + \dots + y_n \vec{w}_n) = y_1 \vec{v}_1 + \dots + y_r \vec{v}_r + y_{r+1} \vec{v}_{r+1} + \dots + y_n \vec{v}_n$$

which implies that $y_1 = \dots = y_n = 0$ since \mathcal{B} is linearly independent. Thus, S is one-to-one by lemma.

Since S is linear and one-to-one, and $\dim \mathbb{V} = \dim \mathbb{V}$ we have that S is also onto by theorem.

Observe that

$$\begin{aligned}(T \circ S \circ T)(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) &= T(S(T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n))) \\ &= T(S(c_1T(\vec{v}_1) + \cdots + c_rT(\vec{v}_r) + \vec{0})) \\ &= T(S(c_1T(\vec{v}_1) + \cdots + c_rT(\vec{v}_r))) \\ &= T(c_1\vec{v}_1 + \cdots + c_r\vec{v}_r) \\ &= T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)\end{aligned}$$

since $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is a basis for $\ker(T)$.