MAT 2384B-Fall 2017-Assignment #5

To be submitted in class on Friday, November 17 at 10:00 AM

Family Name Solution

Given Name

Student Number

- **Please print the assignment and write solutions in the space provided.**
- You can use the back of the pages or additional pages if necessary but be sure to indicate it clearly.
- There are five questions in this assignment.
- You have to answer all the questions.
- Please write your arguments in a clear and readable manner.

Question 1. [12 points] In each case, find the solution to the linear nonhomogeneous system of differential equations: $\vec{y}' = A\vec{y} + \vec{f}(x)$ satisfying the given initial condition:

(a) $\vec{y}' = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix} \vec{y} + \begin{bmatrix} e^x \\ e^{3x} \end{bmatrix}, \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $\vec{y}' = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix}, \quad \vec{y}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

(c) $\vec{y}' = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \vec{y} + \begin{bmatrix} -2x^2+14x+14 \\ -4x^2+18x+23 \end{bmatrix}; \quad \vec{y}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(a) The corresponding homogeneous system is $\vec{y}' = \underbrace{\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}}_{A} \vec{y}$ (H)

$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 + 4 = 0 \Rightarrow (1-\lambda)^2 = -4 = 4i^2$

$\Rightarrow 1-\lambda = \pm 2i \Rightarrow \lambda = 1 \pm 2i$. Take $\lambda = 1+2i$ and find a basis for the corresponding eigenspace E_{1+2i} by solving the system $[A-(1+2i)I \mid \vec{0}]$:

$$\left[\begin{array}{cc|c} 1-(1+2i) & -4 & 0 \\ 1 & 1-(1+2i) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ -2i & -4 & 0 \end{array}\right] \overset{2iR_1+R_2 \to R_2}{\sim}$$

$$\left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$ so $x_2 = t$ is free and $x_1 = 2it$

$E_{1+2i} = \left\{ \begin{bmatrix} 2it \\ t \end{bmatrix} ; t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} t ; t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 2i \\ 1 \end{bmatrix} \right\}$.

So $\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ is a basis for E_{1+2i}. So $\begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(1+2i)x}$ is a complex solution for (H). We find the real and imaginary parts of that solution:

$$\begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(1+2i)x} = \begin{bmatrix} 2i(e^x\cos(2x) + i e^x \sin(2x)) \\ e^x\cos(2x) + i e^x \sin(2x) \end{bmatrix} = \begin{bmatrix} -2e^x \sin(2x) + i\, 2e^x \cos(2x) \\ e^x \cos(2x) + i e^x \sin(2x) \end{bmatrix} =$$

$\begin{bmatrix} -2\sin(2x) \\ \cos(2x) \end{bmatrix} e^x + i \begin{bmatrix} 2\cos(2x) \\ \sin(2x) \end{bmatrix} e^x$. The general solution of (H) is

$$\vec{Y}_H = C_1 \begin{bmatrix} -2e^x \sin(2x) \\ e^x \cos(2x) \end{bmatrix} + C_2 \begin{bmatrix} 2e^x \cos(2x) \\ e^x \sin(2x) \end{bmatrix} = \begin{bmatrix} -2C_1 \sin(2x) + 2C_2 \cos(2x) \\ C_1 \cos(2x) + C_2 \sin(2x) \end{bmatrix} e^x$$

For the particular solution, note that $\vec{f}(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3x}$.

$$\left.\begin{array}{l} \bullet \text{ For } \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^x \text{, choose } \vec{Y}_p = \vec{U} e^x \\ \bullet \text{ For } \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3x} \text{, choose } \vec{Y}_p = \vec{W} e^{3x} \end{array}\right\} \text{ By the sum Rule } \vec{Y}_p = \vec{U} e^x + \vec{W} e^{3x}$$

$\vec{Y}_p' = \vec{U} e^x + 3\vec{W} e^{3x}$. Replace in (NH), we get:

$$\vec{U} e^x + 3\vec{W} e^{3x} = A\left[\vec{U} e^x + \vec{W} e^{3x}\right] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3x} \Rightarrow \begin{cases} A\vec{U} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{U} & (1) \\ A\vec{W} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\vec{W} & (2) \end{cases}$$

$$(1) \Rightarrow (A - I)\vec{U} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \vec{U} = (A - I)^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \end{bmatrix}$$

$$(2) \Rightarrow (A - 3I)\vec{W} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \vec{W} = (A - 3I)^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ -1/8 & -1/4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix}$$

So $\vec{Y}_p = \begin{bmatrix} 0 \\ 1/4 \end{bmatrix} e^x + \begin{bmatrix} -1/2 \\ 1/4 \end{bmatrix} e^{3x} = \begin{bmatrix} -1/2\, e^{3x} \\ \frac{1}{4} e^x + \frac{1}{4} e^{3x} \end{bmatrix}$. The general solution to (NH) is

$$\vec{y} = \vec{Y}_H + \vec{Y}_p = \begin{bmatrix} -2C_1 e^x \sin(2x) + 2C_2 e^x \cos(2x) - \frac{1}{2} e^{3x} \\ C_1 e^x \cos(2x) + C_2 e^x \sin(2x) + \frac{1}{4} e^x + \frac{1}{4} e^{3x} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2C_2 - \frac{1}{2} = 1 \\ C_1 + \frac{1}{4} + \frac{1}{4} = 0 \end{cases} \Rightarrow \begin{array}{l} C_1 = -\frac{1}{2}, \\ C_2 = 3/4 \end{array}$$

The unique solution to (NH) is

$$\vec{y} = \begin{bmatrix} e^x \sin(2x) + \frac{3}{2} e^x \cos(2x) - \frac{1}{2} e^{3x} \\ -\frac{1}{2} e^x \cos(2x) + \frac{3}{4} e^x \sin(2x) + \frac{1}{4} e^x + \frac{1}{4} e^{3x} \end{bmatrix}$$

b) The Corresponding homogeneous system is $\vec{y}' = \underbrace{\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}}_{A} \vec{y}$ (H)

$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow$

$\lambda_1 = 1$, $\lambda_2 = 2$: two distinct eigenvalues

$$E_1 : (A - I \mid \vec{0}) \sim \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \overset{-R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{3} R_1}{\sim} \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$ is free, $x_1 = \frac{2}{3} t \Rightarrow E_1 = \left\{ \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} t ; t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

$\vec{v}_1 = \begin{bmatrix}2\\3\end{bmatrix}$ is a basis for E_1

E_2: $[A-2I\,|\,\vec{0}] \sim \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\frac{1}{3}R_3}{\overset{\frac{1}{2}R_1}{\sim}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \overset{-R_1+R_2\to R_2}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$ is free, $x_1 = t$

$E_2 = \left\{\begin{bmatrix}1\\1\end{bmatrix}t;\ t\in\mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. so $\vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ is a basis for E_2.

The general solution for (H) is $\vec{Y}_H = C_1\begin{bmatrix}2\\3\end{bmatrix}e^{x} + C_2\begin{bmatrix}1\\1\end{bmatrix}e^{2x} = \begin{bmatrix}2C_1e^{x}+C_2e^{2x}\\3C_1e^{x}+C_2e^{2x}\end{bmatrix}$.

For the particular solution of (NH), note that $\vec{f}(x) = \begin{bmatrix}-2\\-2\end{bmatrix}x + \begin{bmatrix}-5\\-3\end{bmatrix}$ is a polynomial of degree 1 (with vector coefficients). So choose $\vec{Y}_p = \vec{U}x + \vec{W}$ for some vectors $\vec{U}$ and $\vec{W}$:

$\vec{Y}_p' = \vec{U}$. Replace in (NH):

$\vec{U} = A(\vec{U}x + \vec{W}) + \begin{bmatrix}-2\\-2\end{bmatrix}x + \begin{bmatrix}-5\\-3\end{bmatrix} \Rightarrow \begin{cases} A\vec{U} + \begin{bmatrix}-2\\-2\end{bmatrix} = \vec{0} & (1) \\ A\vec{W} + \begin{bmatrix}-5\\-3\end{bmatrix} = \vec{U} & (2) \end{cases}$

(1) $\Rightarrow A\vec{U} = \begin{bmatrix}2\\2\end{bmatrix} \Rightarrow \vec{U} = A^{-1}\begin{bmatrix}2\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 & 2\\-3 & 4\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

(2) $\Rightarrow A\vec{W} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}-5\\-3\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix} \Rightarrow \vec{W} = A^{-1}\begin{bmatrix}6\\4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 & 2\\-3 & 4\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

So $\vec{Y}_p = \begin{bmatrix}1\\1\end{bmatrix}x + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}x+1\\x-1\end{bmatrix}$. The general solution of (NH) is $\vec{Y} = \vec{Y}_H + \vec{Y}_p$

$\vec{Y} = \begin{bmatrix}2C_1e^{x}+C_2e^{2x}+x+1\\3C_1e^{x}+C_2e^{2x}+x-1\end{bmatrix}$; $\vec{Y}(0) = \begin{bmatrix}7\\3\end{bmatrix} \Rightarrow \begin{cases}2C_1 + C_2 + 1 = 7\\3C_1 + C_2 - 1 = 3\end{cases} \Rightarrow \begin{cases}2C_1 + C_2 = 6 & (1)\\3C_1 + C_2 = 4 & (2)\end{cases}$

$-(1)+(2) \Rightarrow C_1 = -2$; (1) $\Rightarrow C_2 = 10$. The unique solution is

$$\vec{Y} = \begin{bmatrix}-4e^{x}+10e^{2x}+x+1\\-6e^{x}+10e^{2x}+x-1\end{bmatrix}$$

c) The corresponding homogeneous system is $\vec{Y}' = \underbrace{\begin{bmatrix}2 & -4\\4 & -6\end{bmatrix}}_{A}\vec{Y}$ (H)

$\det(A-\lambda I) = 0 \Rightarrow \det\begin{bmatrix}2-\lambda & -4\\4 & -6-\lambda\end{bmatrix} = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$

is a double real eigenvalue.

E_{-2} : $[A-(-2)I\,|\,\vec{0}] \sim \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_2 = t$ is free, $x_1 = t$

So $E_{-2} = \left\{\begin{bmatrix}t\\t\end{bmatrix};\ t\in\mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ is a basis for E_{-2}.

Next, we look at a generalized eigenvector $\vec{p}$: $[A-(-2I) \mid \vec{v}] \sim$

$$\left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1/4 \\ 0 & 0 & 0 \end{array}\right]$$ $x_2 = t$ is free, $x_1 = t + 1/4$.

Take $t = 0 \Rightarrow x_1 = 1/4$, $x_2 = 0$, $\vec{p} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

The general solution of (H) is $\vec{Y}_H = c_1 \vec{v} e^{-2x} + c_2 (x\vec{v} + \vec{p}) e^{-2x}$

$$\vec{Y}_H = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} x + 1/4 \\ x \end{bmatrix} e^{-2x} = \begin{bmatrix} c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{4} c_2 e^{-2x} \\ c_1 e^{-2x} + c_2 x e^{-2x} \end{bmatrix}$$

For the particular solution, note that $\vec{f} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} x^2 + \begin{bmatrix} 14 \\ 18 \end{bmatrix} x + \begin{bmatrix} 14 \\ 23 \end{bmatrix}$ is a polynomial of degree 2 with vector coefficients, so choose $\vec{Y}_p = \vec{U} x^2 + \vec{W} x + \vec{T}$ for some vectors $\vec{U}, \vec{V}, \vec{T} \in \mathbb{R}^2$.

$\vec{Y}_p' = 2\vec{U} x + \vec{W}$. Replace in (NH):

$$2\vec{U}x + \vec{W} = A(\vec{U}x^2 + \vec{W}x + \vec{T}) + \begin{bmatrix} -2 \\ -4 \end{bmatrix} x^2 + \begin{bmatrix} 14 \\ 18 \end{bmatrix} x + \begin{bmatrix} 14 \\ 23 \end{bmatrix} \Rightarrow \begin{cases} A\vec{U} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \vec{0} & ① \\ A\vec{W} + \begin{bmatrix} 14 \\ 18 \end{bmatrix} = 2\vec{U} & ② \\ A\vec{T} + \begin{bmatrix} 14 \\ 23 \end{bmatrix} = \vec{W} & ③ \end{cases}$$

$$① \Rightarrow \vec{U} = A^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$② \Rightarrow A\vec{W} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 14 \\ 18 \end{bmatrix} = \begin{bmatrix} -12 \\ -18 \end{bmatrix} \Rightarrow \vec{W} = A^{-1} \begin{bmatrix} -12 \\ -18 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -12 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$③ \Rightarrow A\vec{T} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 14 \\ 23 \end{bmatrix} = \begin{bmatrix} -14 \\ -20 \end{bmatrix} \Rightarrow \vec{T} = A^{-1} \begin{bmatrix} -14 \\ -20 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -14 \\ -20 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So $\vec{Y}_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x^2 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x^2 + 1 \\ 3x + 4 \end{bmatrix}$ and the general solution to (NH) is

$$\vec{Y} = \begin{bmatrix} c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{4} c_2 e^{-2x} + x^2 + 1 \\ c_1 e^{-2x} + c_2 x e^{-2x} + 3x + 4 \end{bmatrix}$$

$$\vec{Y}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} c_1 + \frac{1}{4} c_2 + 1 = \\ c_1 + 4 = 5 \end{cases} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -8 \end{array}$$

The unique solution is

$$\vec{Y} = \begin{bmatrix} e^{-2x} - 8x e^{-2x} - 2 e^{-2x} + x^2 + 1 \\ e^{-2x} - 8x e^{-2x} + 3x + 4 \end{bmatrix} = \boxed{\begin{bmatrix} -e^{-2x} - 8x e^{-2x} + x^2 + 1 \\ e^{-2x} - 8x e^{-2x} + 3x + 4 \end{bmatrix}}$$

Question 2. [4 points] Find the Laplace Transform of the following functions:

(a) $f(t) = 2t^4 - 4t^3 + 7t^2 + 3t - 2$

(b) $g(t) = 5e^{-\pi t} + 6\cos(3t)$

(a) $\mathcal{L}\{2t^4 - 4t^3 + 7t^2 + 3t - 2\} = 2\,\frac{(4!)}{s^5} - 4\,\frac{(3!)}{s^4} + 7\,\frac{2!}{s^3} + \frac{3}{s^2} - \frac{2}{s}$

$= \boxed{\frac{48}{s^5} - \frac{24}{s^4} + \frac{14}{s^3} + \frac{3}{s^2} - \frac{2}{s}}$

(b) $\mathcal{L}\{5e^{-\pi t} + 6\cos(3t)\} = 5\mathcal{L}\{e^{-\pi t}\} + 6\mathcal{L}\{\cos(3t)\}$

$= \frac{5}{s-(-\pi)} + 6\left(\frac{s}{s^2+3^2}\right) = \boxed{\frac{5}{s+\pi} + \frac{6s}{s^2+9}}$

Question 3. [4 points] Find the Inverse Laplace Transform of the following functions:

(a) $F(s) = \frac{3s-1}{s^2+s-6}$

(b) $G(s) = \frac{2s+12}{s^2+9}$

(a) $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3s-1}{(s+3)(s-2)}\right\}$ use partial fractions

$$\frac{3s-1}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2} = \frac{(A+B)s - 2A + 3B}{(s+3)(s-2)} \Rightarrow \begin{cases} A+B=3 & ① \\ -2A+3B=-1 & ② \end{cases}$$

$2①+② \Rightarrow 5B = 5 \Rightarrow B = 1$; $① \Rightarrow A = 2$. so

$$\frac{3s-1}{(s+3)(s-2)} = \frac{2}{s+3} + \frac{1}{s-2} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \boxed{2e^{-3t} + e^{2t}}$$

(b) $\mathcal{L}^{-1}\{G(s)\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + 4\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \boxed{2\cos(3t) + 4\sin(3t)}$

Question 4. **[5 points]** Use Laplace Transform to solve the following initial value problem:

$$y'' + 2y' - 3y = 10e^{2t}, \quad y(0) = 0, \; y'(0) = 14.$$

let $Y(s) = \mathcal{L}\{y(t)\}$. Apply the operator $\mathcal{L}$ to the ODE:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 10\,\mathcal{L}\{e^{2t}\} \Rightarrow$$

$$s^2 Y - s y(0) - y'(0) + 2(sY - y(0)) - 3Y = \frac{10}{s-2} \Rightarrow$$

$$\underbrace{(s^2+2s-3)}_{(s-1)(s+3)} Y = 14 + \frac{10}{(s-2)} \Rightarrow Y = \frac{14}{(s-1)(s+3)} + \frac{10}{(s-1)(s-2)(s+3)}$$

we use partial fractions:

- $$\frac{14}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} = \frac{(A+B)s + 3A - B}{(s-1)(s+3)} \Rightarrow \begin{cases} A+B=0 \\ 3A-B=14 \end{cases} \Rightarrow A = 7/2, \; B = -7/2$$

- $$\frac{10}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3} = \frac{A(s^2+s-6) + B(s^2+2s-3) + C(s^2-3s+2)}{(s-1)(s-2)(s+3)}$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ A+2B-3C=0 \\ -6A-3B+2C=10 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -3 & 0 \\ -6 & -3 & 2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 3 & 8 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 20 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Rightarrow A = -5/2, \; B = 2, \; C = 1/2$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 7/2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 7/2\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y(t) = \frac{7}{2}e^{t} - \frac{7}{2}e^{-3t} - \frac{5}{2}e^{t} + 2e^{2t} + \frac{1}{2}e^{-3t} = \boxed{e^{t} - 3e^{-3t} + 2e^{2t}}$$

Question 5. **[4 points]** Consider the following definite integral:

$$J = \int_0^1 \sec^2 x \, dx.$$

1. Use **Gaussian Quadrature of order 4** to estimate the value of J. Round the nodes and the coefficients to 6 decimal places.

2. Use the fundamental Theorem of Calculus to find the value of J rounded to 6 decimal places. Estimate the value of the absolute error |True value-approximate value|.

(1) We start by converting J into the form $\int_{-1}^{1} f(t)\,dt$ using the substitution $x = \frac{b+a}{2} + \frac{b-a}{2} t = \frac{1}{2} + \frac{1}{2} t \Rightarrow dx = \frac{1}{2} dt$

$J = \int_{-1}^{1} \sec^2\left(\frac{1+t}{2}\right) \frac{1}{2} dt$. using Gaussion Quadrature of order 4:

$$\int_{-1}^{1} \sec^2\left(\frac{1+t}{2}\right) \frac{1}{2} dt \simeq w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3) + w_4 f(t_4)$$

$$\frac{0.347855}{2} \sec^2\left(\frac{1-0.861136}{2}\right) + \frac{0.652145}{2} \sec^2\left(\frac{1-0.339981}{2}\right) +$$

$$\frac{0.652145}{2} \sec^2\left(\frac{1+0.339981}{2}\right) + \frac{0.347855}{2} \sec^2\left(\frac{1+0.861136}{2}\right) = \boxed{1.557210}$$

2) $\int_0^1 \sec^2 x\,dx = [\tan x]_0^1 = \tan 1 = 1.557408$

$|\text{True value} - \text{approximate value}| = |1.557408 - 1.557210| = \boxed{0.000198}$