Review: Exponents and Radicals

Def: Let a be a real number and let n be a positive integer. (denoted by $a \in \mathbb{R}$ and $n \in \mathbb{N}$). We define

- $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$
- If $a \neq 0$, we define $a^{-n} = \frac{1}{a^n}$
- $a^0 = 1$, provided that $a \neq 0$.

Def: Let $a > 0$ and let $n \in \mathbb{N}$. We denote the n-th root of a by $\sqrt[n]{a}$. $\sqrt[n]{a}$ is the unique positive real number whose n-th power is a, ie. $(\sqrt[n]{a})^n = a$ (this is the definition.)

Def: Let $a > 0$ and let r be a rational number. Write $r = \frac{p}{q}$ for some integers p, q (denoted by $p, q \in \mathbb{Z}$) with $q \neq 0$.

Define $a^r = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$

Properties of exponents.

Let $a, b > 0$; $r, s \in \mathbb{Q}$, then we have:

- $a^r a^s = a^{r+s}$
- $\frac{a^r}{a^s} = a^{r-s}$
- $(a^r)^s = a^{rs}$
- $(ab)^r = a^r b^r$
- $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

Properties of radical:

Let $a, b > 0$. Let $p, q \in \mathbb{N}$, then:

- $\sqrt[p]{ab} = \sqrt[p]{a}\,\sqrt[p]{b}$
- $\sqrt[p]{\frac{a}{b}} = \frac{\sqrt[p]{a}}{\sqrt[p]{b}}$

Ex. 1 Simplify $(2xy)^{-2} \cdot 2^3 \cdot (yx)^3$

Solution: $(2xy)^{-2} \cdot 2^3 \cdot (yx)^3$

$= \frac{1}{(2xy)^2} \cdot 2^3 \cdot (yx)^3$

$= \frac{1}{2^2 \cdot x^2 y^2} \cdot 2^3 \cdot y^3 \cdot x^3$

$= 2xy$

Ex. 2: Show that for $r \neq 1$,

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}.$$

Solution: We consider

$$(1+r+r^2+\cdots+r^{n-1})(1-r)$$
$$= (1+r+r^2+\cdots+r^{n-1})(1) - (1+r+\cdots+r^{n-1})(r)$$
$$= (1+r+r^2+\cdots+r^{n-1}) - (r+r^2+\cdots+r^n)$$
$$= 1-r^n$$

Therefore $1+r+r^2+\cdots+r^{n-1}$

$$= \frac{1-r^n}{1-r}.$$

Ex. 3. Find the value(s) of a such that

$$x^4 = (x^2+ax+1)(x^2-ax+1).$$

Solution: (Method of determined coefficients).

We expand the right hand side:

$$(x^2+ax+1)(x^2-ax+1)$$
$$= [(x^2+1)+ax][(x^2+1)-ax]$$
$$= (x^2+1)^2 - (ax)^2$$
$$= x^4 + 2x^2 + 1 - a^2x^2$$
$$= x^4 + (2-a^2)x^2 + 1$$

In order that $x^4 = (x^2+ax+1)(x^2-ax+1)$, we should set

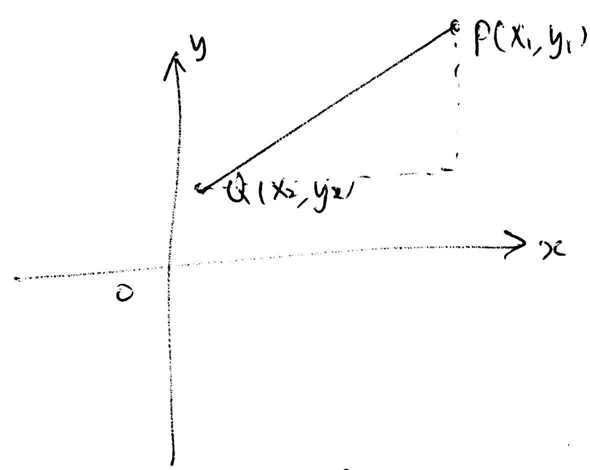
$$2 - a^2 = 0$$
$$a^2 = 2$$
$$a = \sqrt{2} \text{ or } a = -\sqrt{2}.$$

Review: Coordinate Geometry and equations of straight lines.

• Distance formula.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be any two points on the plane. The distance between PQ is given by:

$$|PQ| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$



The formula follows from the Pythagorean Theorem.

• **Direction**

The direction of a line or a line segment is characterized by a number, its slope.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two distinct points on the plane. The slope of the line segment PQ is defined by

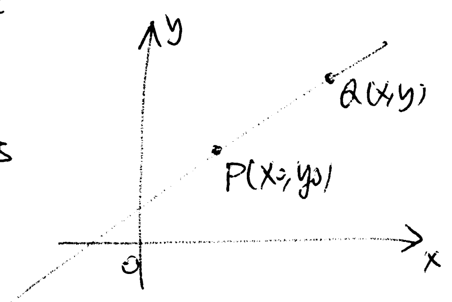
$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Remark: • The slope of a horizontal line segment is 0 (in this case, $y_1 = y_2$)

• The slope of a vertical line segment is undefined (in this case $x_1 = x_2$).

• **Equation of straight line**:

☆ A straight line is completely determined by its direction and a point lying on it.



☆. **Point-slope form**:

Let l be a straight line with slope m passing through the point $P(x_0, y_0)$. The equation of l can be worked out as follow:

Let $Q(x,y)$ be an arbitary point on l, then $\frac{y - y_0}{x - x_0} = m$

$$\therefore \boxed{y - y_0 = m(x - x_0)}$$

• **Slope-interception form**:

Consider a straight line l with slope m and y-intercept c. (i.e. the st. line passes through the point $(0, c)$)

Its equation is given by:

$$\frac{y - c}{x - 0} = m$$

$$\boxed{y = mx + c.}$$

Therefore if we are given an equation of straight line, once we write y as the subject, the coff. of x is the slope of the st line and the constant term is the y-intercept.

Example 1:

Find the slope of the straight line whose equation is $2x+5y-6=0$.

Solution: $2x+5y-6=0$

$$5y=-2x+6$$

$$y=-\frac{2}{5}x+\frac{6}{5}$$

$\therefore$ The slope is $-\frac{2}{5}$.

Example 2: Find the equation of the line passing through the points $(1,2)$, $(-3,8)$.

Solution: The slope of the line is

$$m=\frac{8-2}{-3-1}=-\frac{3}{2}$$

Therefore the equation is:

$$\frac{y-2}{x-1}=m$$

$$\frac{y-2}{x-1}=-\frac{3}{2}$$

$$2(y-2)=-3(x-1)$$

$$2y-4=-3x+3$$

$$3x+2y-7=0.$$

☆ Recall that a point lies on a straight line

$\Leftrightarrow$ The coordinates of the point satisfy the equation of the line.

Therefore, to find the intersection of two lines, we need to solve simultaneous equations.

Example 3:

Find the point of intersection of the two lines $2x+3y+4=0$ and $y=2x-6$.

Solution: We solve the simultaneous equation

$$\begin{cases} 2x+3y+4=0 & \text{——— (1)} \\ y=2x-6 & \text{——— (2)} \end{cases}$$

Sub. (2) into (1), then we have:

$$2x+3(2x-6)+4=0$$

$$8x-14=0$$

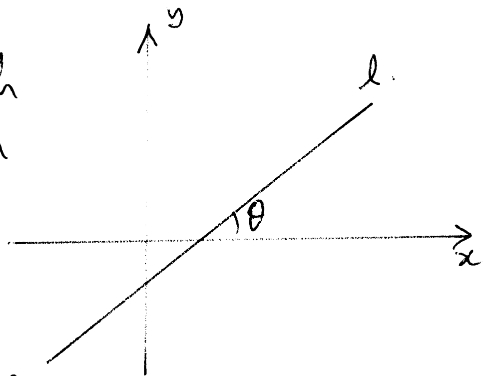
$$x=\frac{7}{4}$$

By (2), $y=2(\frac{7}{4})-6=-\frac{5}{2}$

Therefore the two lines intersect at $(\frac{7}{4},-\frac{5}{2})$.

Parallel and Perpendicular lines

- Let l be a straight line which ~~inter~~ intersect with the x-axis.



The inclination of l is the angle θ between l and the positive x-axis. $(0 \le \theta \le \pi)$

If l is a horizontal line, we define its inclination to be 0.

slope and inclination

$\tan\theta$ = slope, where θ = inclination

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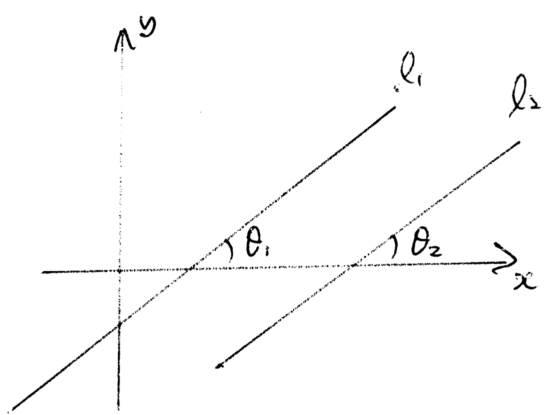
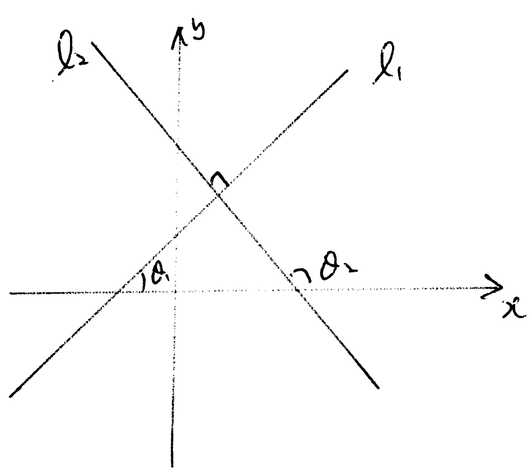




Let m_1 = slope of l_1
m_2 = slope of l_2.

We have:

- $l_1 \,//\, l_2 \iff m_1 = m_2$ provided that the slopes are well-defined.

- $l_1 \perp l_2 \iff m_1 m_2 = -1$ provided that the slopes are well-defined.

Example 4: Find the equation of the straight line through the point $(6,-2)$ that is:

(a) parallel to the line $4x-3y-7=0$.

(b) perpendicular to the line $4x-3y-7=0$.

Solution We simplify $4x-3y-7=0$ and obtain:

$$3y = 4x-7$$

$$y = \frac{4}{3}x - 7$$

$\therefore$ The slope of the st. line is $\frac{4}{3}$.

Let m be the slope of the required st. line

(a) $m = \frac{4}{3}$, so the equation of the line is:

$$\frac{y-(-2)}{x-6} = m$$

$$\frac{y+2}{x-6} = \frac{4}{3}$$

$$4x-3y-30=0.$$

(b). $m \cdot \left(\frac{4}{3}\right) = -1$

$\therefore m = -\frac{3}{4}$

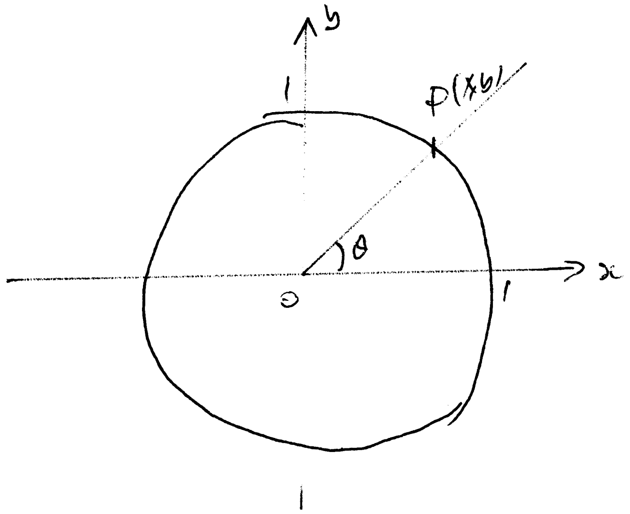
The equation of the st. line is :

$$\frac{y-(-2)}{x-6} = -\frac{3}{4}$$

$$4y+8 = -3x+18$$

$$3x+4y-10=0.$$

§ Trigonometry



Consider the unit circle (ie. circle ~~origin~~ centred at the origin O having radius 1).

We rotate the +ve x-axis about the origin O through an angle θ and suppose that it cuts the ~~circle~~ unit circle at the point P(x,y).

Def

$$\begin{cases} \cos\theta = x \\ \sin\theta = y \\ \tan\theta = \frac{y}{x} \quad (\text{provided that } x \neq 0) \end{cases}$$

We also define

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

provided that they are well-defined.

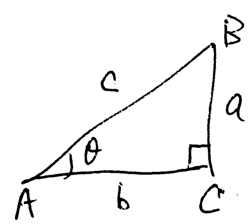
~~In the following diagram~~

- From now on, we use radian measure for an angle, ie.

~~For~~ 1 revolution $= 360^\circ$
$= 2\pi$ (radian)

$\therefore$ $\boxed{\pi = 180^\circ}$

In the diagram, $\triangle ABC$ is a right-angled triangle, then we have :



$$\sin\theta = \frac{a}{c}$$

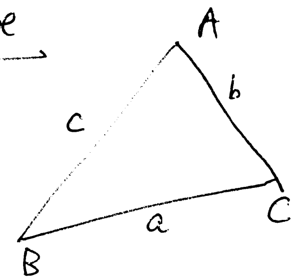
$$\cos\theta = \frac{b}{c}$$

$$\tan\theta = \frac{a}{b}.$$

MATH 1004 Lectures (01)

- Sine and Cosine formulae

On the diagram:



- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(sine formula)

- $c^2 = a^2 + b^2 - 2ab\cos C$ (cosine formula)

- Compound angle formulae

By using the cosine formula and the distance formula, we can derive ~~the~~ compound angle formulae.

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

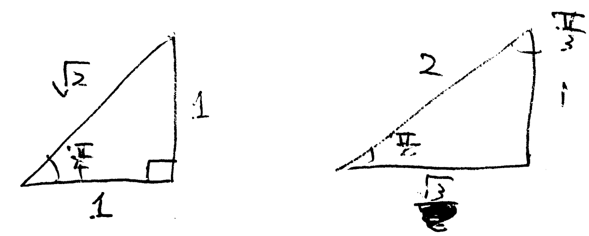
$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Special angles.

By considering the 30-60-90, and the 45-45-90 triangles, we can work out the trigonometric ratios of the special angle:

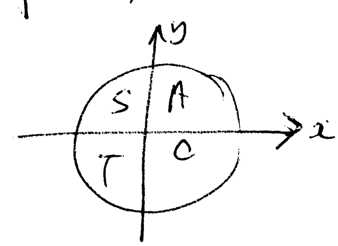
$0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.



$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

$\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$. etc.

- signs of sin, cos, tan.



- ~~trans formula~~

transformation

x-axis Type I. $\boxed{\quad}$ $(\pi \pm \theta)$ $(2\pi \pm \theta)$

y-axis Type. II. $\boxed{\quad}$ $(\frac{\pi}{2} \pm \theta)$ $(\frac{3\pi}{2} \pm \theta)$

For x-axis type
sin → sin
cos → cos
tan → tan

For y-axis type
sin → cos
cos → sin
tan → cot

Pythagoras Theorem: $\sin^2 x + \cos^2 x = 1$.

Ex. 1: Find the value of $\sin \frac{5\pi}{4}$ without using a calculator.

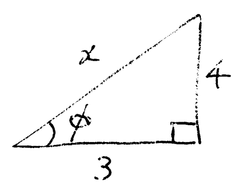
Solution: $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$

$= -\sin\left(\frac{\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}}$.

Ex. 2: Given that $\cot \phi = \frac{3}{4}$ where ϕ is an acute angle, find $\sin\phi$ and $\sec\phi$.

Solution: Consider the diagram:



$x^2 = 3^2 + 4^2$

$\Rightarrow x = 5$.

$\therefore \sin\phi = \frac{4}{5}$ and $\sec\phi = \frac{x}{3} = \frac{5}{3}$.

Alternative:

$\cot\phi = \frac{3}{4} \Rightarrow \tan\phi = \frac{4}{3}$.

$\Rightarrow \frac{\sin\phi}{\cos\phi} = \frac{4}{3}$

$\Rightarrow \sin\phi = \frac{4}{3}\cos\phi$ —— (1)

Since $\sin^2\phi + \cos^2\phi = 1$, we have:

$\left(\frac{4}{3}\cos\phi\right)^2 + \cos^2\phi = 1$

$\frac{16}{9}\cos^2\phi + \cos^2\phi = 1$

$\cancel{16}$ $\frac{25}{9}\cos^2\phi = 1$

$\cos^2\phi = \frac{9}{25}$

$\cos\phi = \frac{3}{5}$ since $0 < \phi < \frac{\pi}{2}$

By (1), $\sin\phi = \frac{4}{3} \cdot \frac{3}{5} = \frac{4}{5}$

$\sec\phi = \frac{1}{\cos\phi} = \frac{5}{3}$.