MAT2122 Multivariable Calculus (Fall 2016)

Final solutions (the total is 80 points)

1 (8 points). Find the distance from the point $P = (0,1,1)$ to the tangent line to the path $\mathbf{c}(t) = (e^{t-1}, \ln t, t^2 - 3t)$ at time $t_0 = 1$.

Solution: The velocity vector of the path $\mathbf{c}$ is

$$\mathbf{c}'(t) = (e^{t-1}, 1/t, 2t-3) \ , \quad ✓$$

whence the velocity at time t_0 (i.e., the direction of the tangent line) is

$$\mathbf{u} = \mathbf{c}'(t_0) = \mathbf{c}'(1) = (1,1,-1) \ . \quad ✓$$

The position at time t_0 is

$$Q = \mathbf{c}(t_0) = \mathbf{c}(1) = (1,0,-2) \ . \quad ✓$$

so that $\overrightarrow{PQ} = (1,-1,-3)$. ✓

The projection of the vector $\overrightarrow{PQ}$ onto vector $\mathbf{u}$ is $\alpha\mathbf{u}$, where

$$\alpha = \frac{\overrightarrow{PQ}\cdot\mathbf{u}}{\|u\|^2} = \frac{1-1+3}{3} = 1 \ , \quad ✓$$

whence the vector

$$\mathbf{v} = \overrightarrow{PQ} - \alpha\mathbf{u} = (1,-1,-3) - (1,1,-1) = (0,-2,-2) \quad ✓$$

is perpendicular to the tangent line. Therefore, the distance from the point P to the tangent line to the path $\mathbf{c}$ at time $t_0 = 1$ is the length of the vector $\mathbf{v}$, i.e., $2\sqrt{2}$. ✓

Absence of minor mistakes. ✓

2 (7 points). Let $g(x,y) = \left(\cos(x+y), e^{xy+1}, \frac{x}{y}\right)$ and $f(u,v,w) = (u\ln v, w^2)$. Find the derivative (by using the chain rule) and the Jacobian of the map $f \circ g$ at the point $(1,-1)$.

Solution: The derivative of the function g is

$$\mathbf{D}g(x,y) = \begin{pmatrix} \big(\cos(x+y)\big)'_x & \big(\cos(x+y)\big)'_y \\ \left(e^{xy+1}\right)'_x & \left(e^{xy+1}\right)'_y \\ \left(\frac{x}{y}\right)'_x & \left(\frac{x}{y}\right)'_y \end{pmatrix} = \begin{pmatrix} -\sin(x+y) & -\sin(x+y) \\ ye^{xy+1} & xe^{xy+1} \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix} ,$$

whence

$$\mathbf{D}g(1,-1) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ -1 & -1 \end{pmatrix} . \quad ✓$$

The derivative of the function f is

$$\mathbf{D}f(u,v,w) = \begin{pmatrix} (u\ln v)'_u & (u\ln v)'_v & (u\ln v)'_w \\ (w^2)'_u & (w^2)'_v & (w^2)'_w \end{pmatrix} = \begin{pmatrix} \ln v & \frac{u}{v} & 0 \\ 0 & 0 & 2w \end{pmatrix} ,$$

so that its value at the point $g(1,-1) = (1,1,-1)$ ✓ is

$$\mathbf{D}f(1,1,-1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$ ✓

Therefore, by the chain rule

$$\mathbf{D}f\circ g(1,-1) = \mathbf{D}f(g(1,-1))\mathbf{D}g(1,-1) = \mathbf{D}f(1,1,-1)\mathbf{D}g(1,-1)$$ ✓

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$$ ✓

whence

$$\mathrm{Jac}\, f\circ g(1,-1) = \det \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} = -4 .$$ ✓

Absence of minor mistakes. ✓

3 (9 points). Find the partial derivatives up to the second order of the function $f(x,y) = e^{x-y}\sin(2(x+y))$. Find the second order Taylor expansion of the function f at the point $(0,0)$ and use it to find an approximate value of $f(0.2,-0.1)$.

Solution: To begin with, $f(0,0) = 0$. Further,

$$f'_x(x,y) = e^{x-y}\sin(2(x+y)) + 2e^{x-y}\cos(2(x+y)) , \qquad f'_x(0,0) = 2 ,$$ ✓

$$f'_y(x,y) = -e^{x-y}\sin(2(x+y)) + 2e^{x-y}\cos(2(x+y)) \qquad f'_y(0,0) = 2 ,$$ ✓

and

$$f''_{xx}(x,y) = -3e^{x-y}\sin(2(x+y)) + 4e^{x-y}\cos(2(x+y)) , \qquad f''_{xx}(0,0) = 4 ,$$ ✓

$$f''_{yy}(x,y) = -3e^{x-y}\sin(2(x+y)) - 4e^{x-y}\cos(2(x+y)) , \qquad f''_{yy}(0,0) = -4 ,$$ ✓

$$f''_{xy}(x,y) = -5e^{x-y}\sin(2(x+y)) , \qquad f''_{xy}(0,0) = 0 .$$ ✓

Therefore, by Taylor's formula

$$f(x,y) \approx f(0,0) + f'_x(0,0)x + f'_y(0,0)y + \frac{1}{2}f''_{xx}(0,0)x^2 + \frac{1}{2}f''_{yy}(0,0)y^2 + f''_{xy}(0,0)xy$$ ✓

$$= 2x + 2y + 2x^2 - 2y^2 ,$$ ✓

whence

$$f(0.2, 0.1) \approx 0.26\ . \text{ ✓}$$

Absence of minor mistakes. ✓

4 (8 points). Find and classify all critical points of the function

$$f(x, y) = x^3 - 3x + 2y^3 - 3y^2 - 12y + 1\ .$$

Solution: Since

$$\nabla f = (3x^2 - 3, 6y^2 - 6y - 12)\ , \text{ ✓}$$

there are 4 critical points obtained by combining the solutions $x_{1,2} = \pm 1$ of the equation $3x^2 - 3 = 0$ and $y_1 = 2, y_2 = -1$ of the equation $6y^2 - 6y - 12 = 0$. ✓ The second derivatives of f are

$$f''_{xx} = 6x\ , \qquad f''_{yy} = 12y - 6\ , \qquad f''_{xy} = 0\ , \text{ ✓}$$

whence the Hessian of f is

$$\text{Hess}\, f = \begin{pmatrix} 6x & 0 \\ 0 & 12y - 6 \end{pmatrix}\ , \text{ ✓}$$

and its determinant is $6x(12y - 6)$. ✓ It is negative at the critical points $(1, -1)$ and $(-1, 2)$, which are therefore saddle points. ✓ If the determinant is negative, then one has to look at the signs of the diagonal entries, which are positive at the point $(1, 2)$ (so that it is a local minimum) and negative at the point $(-1, -1)$ (so that it is a local maximum). ✓

Absence of minor mistakes. ✓

5 (11 points). By using Lagrange multipliers find the absolute minimum and the absolute maximum of the function $f(x, y) = x^2 - y^2$ on the planar region determined by the conditions $x^2 + y^2 \leq 1,\ x + 2y \geq 0,\ 2x - y \geq 0$.

Solution: For finding the global extrema of the function f on the domain D one has to consider (1) critical points of f in the interior of D, (2) critical points of the constrained extremal problem on the interiors of smooth components of the boundary ∂D, (3) singular points of the boundary. ✓

(1) Since $\nabla f = (2x, -2y)$, ✓ the function f has only one critical point $(0, 0)$ which is *not* in the interior of D. ✓

(2) The boundary of D consists of 3 smooth components determined by 3 constraints arising when the corresponding inequality from the definition of D is realized as equality. ✓

(2a) $g(x, y) = x^2 + y^2 = 1$. The corresponding Lagrangian system is then

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \iff \begin{cases} (2x, -2y) = \lambda(2x, 2y) \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} x = \lambda x \\ y = -\lambda y \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} (\lambda - 1)x = 0 \\ (\lambda + 1)y = 0 \\ x^2 + y^2 = 1 \end{cases} \text{ ✓}$$

From the first equation either $\lambda = 1$ (then $y = 0$ from the second equation, and $x = \pm 1$ from the third equation) or $x = 0$ (then $y = \pm 1$ from the third equation, and $\lambda = -1$ from the second equation). It gives 4 critical points $(\pm 1, 0)$ and $(0, \pm 1)$, out of which *only* $(1, 0)$ is in the interior of the corresponding boundary component (as the other two conditions $x + 2y \geq 0$ and $2x - y \geq 0$ are both satisfied as strict inequality). ✓

(2b) $g(x,y) = x + 2y = 0$. The corresponding Lagrangian system is then

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \iff \begin{cases} (2x, -2y) = \lambda(1,2) \\ x + 2y = 0 \end{cases} \iff \begin{cases} x = \lambda/2 \\ y = -\lambda \\ x + 2y = 0 \end{cases}$$

It has the only solution $x = y = \lambda = 0$. The arising critical point $(0,0)$ is *not* in the interior of the boundary component determined by the equation $x + 2y = 0$ as at this point condition $2x - y \geq 0$ is satisfied with equality. [This critical point can also be found directly by noticing that along the line $x + 2y = 0$ the target function f takes the form $f(x) = x^2 - y^2 = x^2 - (x/2)^2 = 3x^2/4$.] ✓

(2c) $g(x,y) = 2x - y = 0$. In the same way as in (2b) above one obtains a critical point $(0,0)$ which is *not* in the interior of this boundary component. ✓

(3) There are 3 singular points of pairwise intersections of boundary components. They are $(0,0)$ (the intersection of the lines $2x - y = 0$ and $x + 2y = 0$), and the intersection points $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$ of these lines with the circle $x^2 + y^2 = 1$. ✓

Altogether there are 4 points susceptible to be the points of global extrema of the function f on the domain D. By evaluating the function f at these points, we find

$$f(1,0) = 1\,, \qquad f(0,0) = 0\,, \qquad f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = -\frac{3}{5}\,, \qquad f\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \frac{3}{5}\,,$$

whence the global maximum equal to 1 is attained at the point $(1,0)$, and the global minimum equal to $-3/5$ is attained at the point $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$. ✓

Absence of minor mistakes. ✓

6 (8 points). Find the average value of the function $f(x,y) = x^2 y$ on the planar domain bounded by the graphs of the functions $y = x^2,\ y = 2x - 1,\ y = -2x - 1$.

Solution: The domain D is bounded by the graphs of 3 functions, whose pairwise intersections are the points obtained by solving the equations

$$\begin{aligned} x^2 &= 2x - 1 &&\implies x = 1 &&\implies y = x^2 = 2x - 1 = 1\,, \\ x^2 &= -2x - 1 &&\implies x = -1 &&\implies y = x^2 = -2x - 1 = 1\,, \\ 2x - 1 &= -2x - 1 &&\implies x = 0 &&\implies y = 2x - 1 = -2x - 1 = -1\,. \end{aligned}$$ ✓

Therefore, the projection of the domain D onto the horizontal line is the interval $[-1,1]$, ✓ and D can be described as

$$D = \{(x,y) : -1 \leq x \leq 1, \phi(x) \leq y \leq x^2\}\,,$$

where the function ϕ is defined as

$$\phi(x) = \begin{cases} -2x - 1\,, & -1 \leq x \leq 0\,, \\ 2x - 1\,, & 0 \leq x \leq 1\,. \end{cases}$$ ✓

The average value of f is

$$\overline{f} = \frac{I}{\text{area } D}\,,$$ ✓

where (because of the symmetry of the domain D and of the function x with respect to the reflection $x \mapsto -x$)

$$\begin{aligned} I &= \iint_D f(x,y)\,dxdy = 2\int_0^1 \int_{2x-1}^{x^2} x^2 y\,dydx = \int_0^1 x^2\big(x^4 - (2x-1)^2\big)\,dx \\ &= \int_0^1 (x^6 - 4x^4 + 4x^3 - x^2)\,dx = \frac{1}{7} - \frac{4}{5} + 1 - \frac{1}{3} = \frac{1}{105}\,, \quad \checkmark \end{aligned}$$

and

$$\text{area}\, D = \int_{-1}^1 \big(x^2 - \phi(x)\big)\,dx = 2\int_0^1 (x^2 - 2x + 1)\,dx = 2\int_0^1 t^2\,dt = \frac{2}{3}\,, \quad \checkmark$$

whence

$$\overline{f} = \frac{\frac{1}{105}}{\frac{2}{3}} = \frac{1}{70}\,. \quad \checkmark$$

Absence of minor mistakes. ✓

7 (9 points). Find the center of mass of the solid "bowl" W of constant density determined by the conditions $1 \le x^2 + y^2 + z^2 \le 4$ and $z \ge 0$.

Solution: Let the coordinates of the center of mass be (x_0, y_0, z_0). Then $x_0 = y_0 = 0$ by rotational symmetry of W, ✓ whereas

$$z_0 = \frac{\iiint_W z\,dV}{\text{vol}\, W}\,, \quad \checkmark$$

where

$$\text{vol}\, W = \frac{1}{2}\cdot\frac{4}{3}\pi(2^3 - 1^3) = \frac{14}{3}\pi\,. \quad \checkmark$$

As for the numerator, in spherical coordinates ✓

$$\begin{aligned} \iiint_W z\,dV &= \int_1^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \rho\cos\phi\cdot\rho^2\sin\phi\,d\phi \quad \checkmark \\ &= \left[\int_1^2 \rho^3\,d\rho\right]\cdot\left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^{\pi/2} \cos\phi\sin\phi\,d\phi\right] \quad \checkmark \\ &= \frac{15}{4}\cdot 2\pi\cdot\frac{1}{2} = \frac{15}{4}\pi\,, \quad \checkmark \end{aligned}$$

whence

$$z_0 = \frac{\frac{15}{4}\pi}{\frac{14}{3}\pi} = \frac{45}{56}\,. \quad \checkmark$$

Absence of minor mistakes. ✓

8 (8 points). Verify Gauss' formula for the cube $W = [0,1]\times[0,1]\times[0,1]$ and the vector field $\mathbf{F}(x,y,z) = (x^2y^2, xyz, z^3)$.

Solution: The divergence of the vector field $\mathbf{F}$ is

$$\begin{aligned} \text{div}\,\mathbf{F} &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \checkmark \\ &= 2xy^2 + xz + 3z^2\,. \quad \checkmark \end{aligned}$$

By Gauss' formula

$$I = \iiint_W \operatorname{div} \mathbf{F}\, dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}\ , \quad ✓$$

whence

$$I = \iiint_W (2xy^2 + xz + 3z^2)\, dV = 2 \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{4} + 1 = \frac{19}{12}\ . \quad ✓$$

In order to find I as a surface integral notice that ∂W consists of 6 square faces

$$\begin{aligned} S_1 &= \{(x,y,z) : x, y \in [0,1], z = 0\}\ , \\ S_2 &= \{(x,y,z) : x, y \in [0,1], z = 1\}\ , \\ S_3 &= \{(x,y,z) : x, z \in [0,1], y = 0\}\ , \quad ✓ \\ S_4 &= \{(x,y,z) : x, z \in [0,1], y = 1\}\ , \\ S_5 &= \{(x,y,z) : y, z \in [0,1], x = 0\}\ , \\ S_6 &= \{(x,y,z) : y, z \in [0,1], x = 1\} \end{aligned}$$

with the outward normals $\mathbf{n}_i$ equal to the vectors $-\mathbf{k}, \mathbf{k}, -\mathbf{j}, \mathbf{j}, -\mathbf{i}, \mathbf{i}$, respectively. Therefore,

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \sum_{i=1}^{6} \iint_{S_i} \mathbf{F} \cdot \mathbf{n}_i\, dS \quad ✓$$

with

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS &= -\int_0^1 \int_0^1 z^3\, dx\, dy \,\bigg|_{z=0} = 0\ , \\ \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS &= \int_0^1 \int_0^1 z^3\, dx\, dy \,\bigg|_{z=1} = 1\ , \\ \iint_{S_3} \mathbf{F} \cdot \mathbf{n}_3\, dS &= -\int_0^1 \int_0^1 xyz\, dx\, dz \,\bigg|_{y=0} = 0\ , \quad ✓ \\ \iint_{S_4} \mathbf{F} \cdot \mathbf{n}_4\, dS &= \int_0^1 \int_0^1 xyz\, dx\, dz \,\bigg|_{y=1} = \frac{1}{4}\ , \\ \iint_{S_5} \mathbf{F} \cdot \mathbf{n}_5\, dS &= -\int_0^1 \int_0^1 x^2y^2\, dy\, dz \,\bigg|_{x=0} = 0\ , \\ \iint_{S_6} \mathbf{F} \cdot \mathbf{n}_6\, dS &= \int_0^1 \int_0^1 x^2y^2\, dy\, dz \,\bigg|_{x=1} = \frac{1}{3}\ , \end{aligned}$$

whence again

$$I = 1 + \frac{1}{4} + \frac{1}{3} = \frac{19}{12}\ .$$

Absence of minor mistakes. ✓

9 (6 points). Use Green's formula to find the area of the planar region D bounded by the curve $\mathbf{c}(t) = (\sin 2t, \sin t),\ 0 \le t \le \pi$.

Solution: The parametrization of $\mathbf{c}$ is counterclockwise, ✓ so that

$$\begin{aligned}
\text{area } D &= \frac{1}{2}\int_{\mathbf{c}} (-y\,dx + x\,dy) \quad \checkmark \\
&= \frac{1}{2}\int_0^{\pi} (-y(t), x(t)) \cdot \mathbf{c}'(t)\,dt = \frac{1}{2}\int_0^{\pi} (-\sin t, \sin 2t) \cdot (2\cos 2t, \cos t)\,dt \quad \checkmark \\
&= \int_0^{\pi} (-\sin t \cos 2t + \sin t \cos^2 t)\,dt = \int_0^{\pi} (\cos^2 t - \cos 2t)\sin t\,dt = \int_0^{\pi} (1 - \cos^2 t)\sin t\,dt \quad \checkmark \\
&= \left[\frac{1}{3}\cos^3 t - \cos t\right]\Bigg|_{t=0}^{t=\pi} = \frac{4}{3}\,. \quad \checkmark
\end{aligned}$$

Absence of minor mistakes. ✓

10 (6 points). Determine whether the vector field $\mathbf{F}(x, y, z) = (2xyz, x^2 z, x^2 y)$ is conservative. If yes, then find an appropriate potential.

Solution: A vector field is conservative if and only if its curl vanishes. Since

$$\begin{aligned}
\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},\right) \quad \checkmark \\
&= (x^2 - x^2, 2xy - 2xy, 2xz - 2xz) = 0\,, \quad \checkmark
\end{aligned}$$

the field $\mathbf{F}$ is indeed conservative. Its potential f must satisfy the relation $\nabla f = \mathbf{F}$, i.e.,

$$\begin{aligned}
f'_x &= 2xyz\,, \\
f'_y &= x^2 z\,, \quad \checkmark \\
f'_z &= x^2 y\,,
\end{aligned}$$

whence

$$\begin{aligned}
f(x, y, z) &= x^2 yz + h_1(y, z)\,, \\
f(x, y, z) &= x^2 yz + h_2(x, z)\,, \quad \checkmark \\
f(x, y, z) &= x^2 yz + h_3(x, y)\,,
\end{aligned}$$

so that $f(x, y, z) = x^2 yz$ ✓ is a potential of the vector field $\mathbf{F}$.

Absence of minor mistakes. ✓