uOttawa

MAT2122, Midterm test

Fall 2015.

Instructor:
Mohammad Bardestani.

Family Name: ______________________

First Name: Solution

Student number: ______________________

- Time: 80 min.
- Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted. The use of a calculator with any of these banned capabilities is considered **academic fraud**.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.
- There are five questions worth a total of 30 marks.

Question 1:**[7 points]** Find and classify all critical points of the function

$$f(x,y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 - \frac{3}{2}x^2 - \frac{1}{2}y^2 + 2x + 6y + 2.$$

First we find critical points of $f(x,y)$, i.e., we find all $(x_0, y_0) \in \mathbb{R}^2$ s.t $\nabla f(x_0, y_0) = (0,0)$ .5 . We have

$$\nabla f(x,y) = (x^2 - 3x + 2, -y^2 - y + 6) \quad .5$$

Hence we should solve $\begin{cases} x^2 - 3x + 2 = 0 \\ -y^2 - y + 6 = 0 \end{cases}$. We have

$\begin{cases} x^2 - 3x + 2 = (x-1)(x-2) = 0 \\ -(y^2 + y - 6) = -y^2 - y + 6 = -(y+3)(y-2) = 0 \end{cases}$. Therefore critical points of $f(x,y)$ are $\boxed{(1,-3), (1,2), (2,-3), (2,2)}$ 1

We now classify these critical points using the second derivative test. We have

$$D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2x-3 & 0 \\ 0 & -2y-1 \end{pmatrix} \quad 1$$

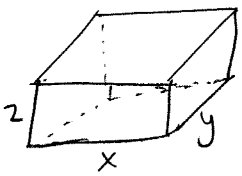
- Point $(1,-3)$. $D^2 f(1,-3) = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow \det D^2 f(1,-3) < 0$ and so $(1,-3)$ is a saddle point. 1
- Point $(1,2)$. $D^2 f(1,2) = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix} \Rightarrow \det D^2 f(1,2) = 5 > 0$ and $\frac{\partial^2 f}{\partial x^2}(1,2) = -1 < 0$ and so $(1,2)$ is a local maximum. 1
- Point $(2,-3)$. $D^2 f(2,-3) = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow \det D^2 f(2,-3) = 5 > 0$ and $\frac{\partial^2 f}{\partial x^2}(2,-3) = 1 > 0$ and so $(2,-3)$ is a local minimum. 1

•) Point $(2,2)$: $D^2f(2,2) = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix} \Rightarrow \det D^2f(2,2) < 0$ and so $(2,2)$ is a saddle point. 1

Question 2: **[6 points]** Assume that among all rectangular boxes with fixed surface area of 10m^2 there is a box of largest possible volume. Find its volume.

If x, y, z are the length of the sides, $x \geq 0, y \geq 0, z \geq 0$ respectively and the volume is $f(x,y,z) = xyz$ 1.

the Constraint is $2(xy + yz + xz) = 10$, that is $xy + xz + yz = 5$ 1.



set $g(x,y,z) = xy + xz + yz$ and let $S = \{(x,y,z) : g(x,y,z) = 5\}$ and so we have to maximize $f|_S$. Thus, by Lagrange multiplier we have to find $(x_0, y_0, z_0) \in S$ and $\lambda \in \mathbb{R}$ s.t

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \quad 1$$

and so we should solve

$$\begin{cases} yz = \lambda(y+z) & \text{I} \\ xz = \lambda(x+z) & \text{II} \\ xy = \lambda(x+y) & \text{III} \\ xy + xz + yz = 5 & \text{IV} \end{cases} \quad 1$$

First notice that $x \neq 0$ because $x = 0$ implies $yz = 5$ by IV and we get $\lambda y = 0$ by III so that $\lambda = 0$ and so by I we get $yz = 0$ which is a contradiction since $yz = 5$. Similarly $y \neq 0$ and $z \neq 0$. Since $y + z \neq 0$ and $x + z \neq 0$ then by I and II we get

$$\frac{yz}{y+z} = \frac{xz}{x+z} \Rightarrow \frac{y}{y+z} = \frac{x}{x+z} \Rightarrow \boxed{x = y}$$

Similarly $y = z$, and so by (IV) we obtain

$$3x^2 = 5 \implies \boxed{x = \sqrt{5/3}} \quad 1$$

thus $x = y = z = \sqrt{5/3}$ and the maximum of $f|_S$ is $\left(\sqrt{5/3}\right)^3$. 1

Question 3:

Part A) **[3 points]** Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable function such that $f(2x, y^2) = f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Show that $\nabla f(1, 1) = 2\nabla f(2, 1)$.

Let $g(x,y) = (2x, y^2)$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and so by $f(2x,y^2) = f(x,y)$ we get $f(x,y) = f \circ g(x,y)$. Hence by the chain rule

$$(*) \quad Df(x,y) = Df(g(x,y))\, Dg(x,y). \quad 1$$

But $Dg(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2y \end{bmatrix}$ 1. Hence at $(1,1)$ we get from $(*)$

$$\nabla f(1,1) = \left[\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial y}(1,1)\right] = \left[\frac{\partial f}{\partial x}(g(1,1)) \quad \frac{\partial f}{\partial y}(g(1,1))\right] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \left[2\frac{\partial f}{\partial x}(2,1), 2\frac{\partial f}{\partial y}(2,1)\right]$$

1

$$= 2\nabla f(2,1)$$

Part B) **[3 points]** At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

Set $F(x,y,z) = z^2 + x^2 - y$. Hence the Paraboloid $S = \{(x,y,z) : y = x^2 + z^2\}$ is indeed the level surface of $F(x,y,z) = 0$. The equation of the tangent plane to the level surface S at the Point $(x_0, y_0, z_0) \in S$ is $\nabla F(x_0,y_0,z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$.

Hence the normal vector of the tangent plane at (x_0, y_0, z_0) is $(2x_0, -1, 2z_0)$ 1. Moreove notice that the normal vector of $x + 2y + 3z = 1$ is $(1,2,3)$. therefore we should find all $(x_0,y_0,z_0) \in S$ s.t for some λ we have $\lambda(1,2,3) = \nabla F(x_0,y_0,z_0) = (2x_0, -1, 2z_0)$ 1 so we should solve

$$\begin{cases} \lambda = 2x_0 \\ 2\lambda = -1 \\ 3\lambda = 2z_0 \\ x_0^2 + z_0^2 = y_0 \end{cases} \Rightarrow \lambda = -\tfrac{1}{2} \Rightarrow x_0 = -\tfrac{1}{4},\ z_0 = -\tfrac{3}{4}.$$

$$y_0 = 5/8$$

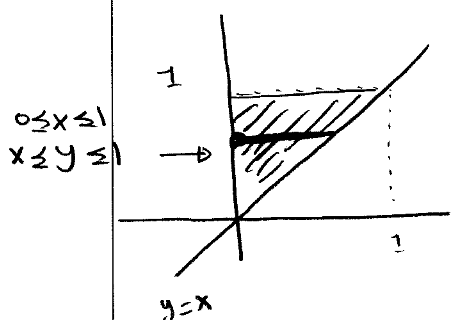
So the tangent plane at $(-1/4, 5/8, -3/4)$ ① is parallel to $x + 2y + 3z = 1$

Question 4:

Part A) [**3 points**] Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_x^1 e^{x/y}\, dy\, dx.$$

Notice that in $I = \int_0^1 \int_x^1 e^{x/y}\, dy\, dx$ we have $0 \le x \le 1$ and $x \le y \le 1$.

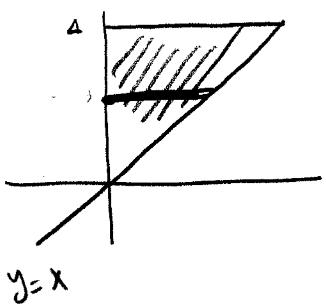


Hence $\int_0^1 \int_x^1 e^{x/y} dy\, dx = \int_0^1 \int_0^y e^{x/y}\, dx\, dy$ (1)

$= \int_0^1 \left[y e^{x/y} \Big|_0^y \right] dy = \int_0^1 (ye - y)\, dy$ (1)

$= \frac{1}{2}(e-1)$. (1)

Part B) [**3 points**] Evaluate $\iint_D y^2 e^{xy}\, dA$ where D is a region bounded by $y = x$, $y = 4$ and $x = 0$.



Hence $\iint_D y^2 e^{xy} dA = \int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy$ (1)

But $\int_0^y y^2 e^{xy}\, dx = y e^{xy} \Big|_0^y = y e^{y^2} - y$ (1)

therefore

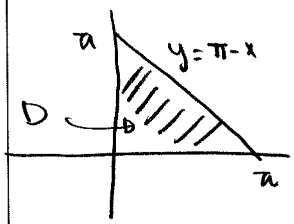
$\iint_D y^2 e^{xy} dA = \int_0^4 (y e^{y^2} - y)\, dy = \frac{1}{2} e^{y^2} - \frac{y^2}{2} \Big|_0^4 = \frac{1}{2} e^{16} - \frac{17}{2}$ (1)

$\boxed{\frac{1}{2} e^{16} - \frac{17}{2} = 4443046.76}$

Question 5: [5 points] Evaluate $\iiint_E \sin y \, dV$ where E lies below the plane $z = x$ and above the triangular region with vertices $(0,0,0)$, $(\pi,0,0)$ and $(0,\pi,0)$.

Here $E = \{(x,y,z) : (x,y) \in D$ and $0 \le z \le x\}$ where

$D = \{(x,y) : 0 \le x \le \pi, \ 0 \le y \le \pi - x\}$ 1



Hence

$$I = \iiint_E \sin y \, dV = \int_0^{\pi} \int_0^{\pi - x} \int_0^{x} \sin y \, dz\, dy\, dx$$ 1

But $\int_0^{x} \sin y \, dz = x \sin y \Rightarrow I = \int_0^{\pi} \int_0^{\pi - x} x \sin y \, dy$ 1

But $\int_0^{\pi - x} x \sin y \, dy = -x \cos y \Big|_0^{\pi - x} = x - x\cos(\pi - x))$

$= x + x \cos x.$ 1

therefore

$$I = \int_0^{\pi} (x + x\cos x)\, dx = \pi^2/2 - 2$$ 1