Math 136

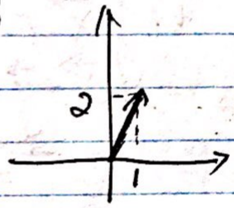
Lectures 1: Vectors in $\mathbb{R}^n$

Def^n: The set $\mathbb{R}^n$ is defined by $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_i \in \mathbb{R} \right\}$

Ex: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ $\quad \begin{bmatrix} 3609601 \\ \sqrt{2} \\ \pi \end{bmatrix} \in \mathbb{R}^3$

Sometimes we will view a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ as the point $(x_1, x_2 \cdots x_n)$ in n-dimentional space.

Example: The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ can be viewed as the point $(1, 2)$



tail at the orgin

Arrows ~~point~~ at the ~~point~~ point

Can't move the vector

Applications: Canadian Economy: $\mathbb{R}^{1500}$

Biology: $\mathbb{R}^{10}$ where the entries are population in a given age class

$$\begin{bmatrix} \text{\# of ppl aged 0~10} \\ \text{\# of pple aged 11~20} \\ \vdots \\ \text{\# of ppl aged 91~100} \end{bmatrix}$$

Def^n Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $t \in \mathbb{R}$, we define addition by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

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we define scalar multiplication

$$t\vec{x} = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix}$$

Example : ① $3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$

② $\pi\begin{bmatrix} e \\ \sqrt{2} \\ 1 \end{bmatrix} + 96059601\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi e \\ \pi\sqrt{2} \\ \pi \end{bmatrix}$

Theorem 1.1.1

let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $s, t \in \mathbb{R}$,

V1) $\vec{x} + \vec{y} \in \mathbb{R}^n$

☆ V2) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$

☆ V3) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

V4) There exists $\vec{0} \in \mathbb{R}^n$ s.t. $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

V5) $\forall \vec{x} \in \mathbb{R}^n$, $\exists (-\vec{x}) \in \mathbb{R}^n$ s.t. $(-\vec{x}) + \vec{x} = \vec{0}$

V6) $t\vec{x} \in \mathbb{R}^n$

☆ V7) $s(t\vec{x}) = (st)\vec{x}$

☆ V8) $(s+t)\vec{x} = s\vec{x} + t\vec{x}$

☆ V9) $s(\vec{x} + \vec{y}) = s\vec{x} + s\vec{y}$

☆ V10) $1\vec{x} = \vec{x}$

$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is called zero ~~to~~ vectors

V2, 3, 7, 8, 9, 10 are operations of addiction and scalar multiplication

V1, 4, 5, 6 are about the relationship between the operations and the set $\mathbb{R}^n$

V1 and V6 show that $\mathbb{R}^n$ is closed under linear combinations.

i.e. if $\vec{v}_1, \cdots \vec{v}_k \in \mathbb{R}^n$ then $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots + c_k\vec{v}_k \in \mathbb{R}^n$ for any $c_1, \cdots c_k \in \mathbb{R}$

(there are sets which are not closed under linear combinations)

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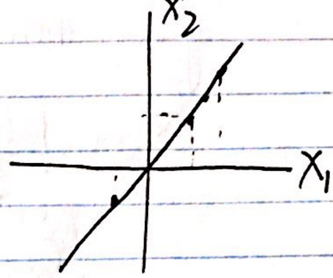
Lecture 2 Spanning

Example: what is a geometric interpretation of

$$S=\left\{ s\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, s\in\mathbb{R}\right\}$$



Every vector in S has the form

$\vec{x}=s\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}s\\s\end{bmatrix}$

$\Rightarrow$ the ~~plant~~ point $(s,s) \Rightarrow x_2=x_1$, Hence the set of all such points is the line $x_2=x_1$

Example: what is the geometric interpretation of

$$T=\left\{ s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix} \,\middle|\, s,t\in\mathbb{R}\right\}$$

Sol^n: T contains vectors of the form $s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}s\\t\\0\end{bmatrix}$

$\Rightarrow (s,t,0)$ the x_1x_2-plane.

We sometimes will write such sets as vector equations
For $S=\left\{ s\begin{bmatrix}1\\1\end{bmatrix} \,\middle|\, s\in\mathbb{R}\right\}$ we'll write its vector equation as
$\vec{x}=s\begin{bmatrix}1\\1\end{bmatrix},\ s\in\mathbb{R}$

infinitely many vectors.

A vector equation for the set T is $\vec{x}=s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}, \boxed{s,t\in\mathbb{R}}$

$\vec{x}=s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}$
represents a single linear combination

Def^n: Let $B=\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$, we define the <u>span of B</u> by

$$\text{Span}\, B=\{t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k \mid t_1,t_2\cdots t_k\in\mathbb{R}\}$$

We say that S is spaned by B, and that B is a spanning set for S

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Example: describe Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ geometrically.

Sol$^{\underline{n}}$: By definition, the spanned set has vector equation

$$\vec{x} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix}; \quad c_1, c_2 \in \mathbb{R}$$

$$= \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad c_1, c_2 \in \mathbb{R} \Rightarrow (c_1, c_2), \text{ So it is } \mathbb{R}^2$$

Example: ~~Q~~ Is the $\vec{x} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ in $S = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right\}$?

Sol$^{\underline{n}}$: We need to determine if there exits $c, d \in \mathbb{R}$ s.t.

$$\begin{bmatrix}1\\2\\1\end{bmatrix} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}-2\\1\\-2\end{bmatrix} \quad \Bigg| \quad \begin{bmatrix}c-2d\\c+d\\c-2d\end{bmatrix}$$

$1 = c - 2d$

$2 = c + d$ $\qquad c = \frac{5}{3} \quad d = \frac{1}{3}$ Hence $\vec{x} \in$ span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right\}$

~~$4 = c$~~ $1 = c - 2d$

Example: Let $B = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\3\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$

Is $\vec{x} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ in Span B

Sol$^{\underline{n}}$: Consider $c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\0\\3\end{bmatrix} + c_3\begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}c_1 - 2c_2 + 2c_3\\0\\c_1 + 3c_2 + 2c_3\end{bmatrix}$

$1 = c_1 - 2c_2 + 2c_3$

$2 = 0$ $\qquad$ no solutions, so $\vec{x} \notin B$

$1 = c_1 + 3c_2 + 2c_3$

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Example: Find a geometric interpretation and a simplified vector equation for each set.

a) Span$\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}-2\\4\\-2\end{bmatrix}\right\}$

Sol^n: A vector equation is $\vec{x} = s\begin{bmatrix}1\\-2\\1\end{bmatrix} + t\begin{bmatrix}-2\\4\\-2\end{bmatrix}$ $s,t \in \mathbb{R}$

$= s\begin{bmatrix}1\\-2\\1\end{bmatrix} - 2t\begin{bmatrix}1\\-2\\1\end{bmatrix}$ $s,t \in \mathbb{R}$

$= (s-2t)\begin{bmatrix}1\\-2\\1\end{bmatrix}$ $s,t \in \mathbb{R}$

$= a\begin{bmatrix}1\\-2\\1\end{bmatrix}$ $a \in \mathbb{R}$

a line through the orgin

b). Span$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\2\\5\end{bmatrix}\right\}$

Sol^n: A vector equation is $\vec{x} = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}-1\\2\\4\end{bmatrix} + c\begin{bmatrix}0\\2\\5\end{bmatrix}$ $a,b,c \in \mathbb{R}$

$= a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}-1\\2\\4\end{bmatrix} + c\left(\begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\2\\4\end{bmatrix}\right)$ $a,b,c \in \mathbb{R}$

take $s = a+c$, $t = b+c$.

$= s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}-1\\2\\4\end{bmatrix}$, $s,t \in \mathbb{R}$

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It's not unique

geometric interpretation:
how many possible scalar multiples of $\vec{v}$

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