Test 3, SOLUTIONS MATH 1104 , Fall, 2017

Last name: First name: Student no.:

This test has 7 extra marks! The exceeded marks (more than 40) will NOT be transferred to other tests.

1-[5 marks]: Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. You are given that $\mathbf{x}$ is an eigenvector of A. What is the corresponding eigenvalue?

(a) 1 **(b)** 2 **(c)** 3 **(d)** -3 **(e)** -2 **(f)** None

Answer: **(c)**

2-[5 marks]: The eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are:

(a) 1 , 2 **(b)** -2 , 6 **(c)** -3 , 4 **(d)** 2 only **(e)** -1 , 5 **(f)** None

Answer: **(e)**

3-[5 marks]: What is the standard form $a + bi$ of the complex number $\dfrac{\overline{1-5i}}{(2+2i)(5i)}$?

(a) $2-3i$ **(b)** $2+3i$ **(c)** $-2+3i$ **(d)** $3-2i$ **(e)** $-3+2i$ **(f)** None

Answer: **(a)**

4-[8+5 marks]: Given that $\lambda_1 = \lambda_2 = 1$ and $\lambda_2 = 10$ are the eigenvalues for $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

a: Find all associated eigenvectors and basis for the eigenspaces.

b: Is A diagonalizable? if yes, find matrix P that diagonalizes A, if not justify your answer.

Solution

a: For $\lambda_1 = 1$ the augmented matrix of $(A - I_3)\mathbf{x} = \mathbf{0}$ is, $\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3'=R_3+(\frac{-1}{2})R_1]{R_2'=R_2-R_1}$ $\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. So $x_3 = r$, $x_2 = s$ and $x_1 = (\frac{-1}{2})r - s$. Thus, $\mathbf{x}_1 = \begin{bmatrix} \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. OR: $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are basis vectors.

For $\lambda_3 = 10$ the augmented matrix is, $\left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & 2 & -8 & 0 \\ 4 & -5 & 2 & 0 \\ -5 & 4 & 2 & 0 \end{array}\right] \begin{array}{l} R_1' = (\frac{1}{2})R_1 \\ R_2' = R_2 + (-2)R_1 \\ R_3' = R_3 + (\frac{5}{2})R_1 \end{array} \longrightarrow$ $\left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 9 & -18 & 0 \end{array}\right] \xrightarrow[R_3'=R_3+R_2]{R_2'=(\frac{1}{9})R_2} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. So $x_3 = r$, $x_2 = 2r$ and $x_1 = 2r$. Thus, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is a basis vector.

b: A is diagonalizable because vectors $x_1, x_2 x_3$ are linearly independent. Then $P = [\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}]$ where $\mathbf{x_i}$s are as above.

5-[12 marks]: Let $W = span \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} \right\}$. Find bases for W and $W^{\perp}$.

Solutions: $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = R$ (R is in RREF!). So a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$

Also $R\mathbf{x} = \mathbf{0} \Longrightarrow W^{\perp} = span \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

6-[2+5 marks]: Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$.

a: Show that $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ form a set of orthogonal vectors.

b: Express $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ as a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$.

Solution:

a: $\mathbf{v_1}.\mathbf{v_2} = \mathbf{v_2}.\mathbf{v_3} = \mathbf{v_1}.\mathbf{v_3} = 0$. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal vectors.

b: Put $c_1 = \frac{\mathbf{v}_1.\mathbf{x}}{\mathbf{v}_1.\mathbf{v}_1} = \frac{5}{2}$, $c_2 = \frac{\mathbf{v}_2.\mathbf{x}}{\mathbf{v}_2.\mathbf{v}_2} = \frac{1}{2}$, $c_3 = \frac{\mathbf{v}_3.\mathbf{x}}{\mathbf{v}_3.\mathbf{v}_3} = \frac{1}{6}$ Then $\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$.