Class Notes Jan 22, MAT1322B - Calculus II

Work - continued

Example: We pump gasoline out of an underground gasoline tank of the shape of a horizontal cylinder. The length of the cylinder is 12m, the radius of the side wall which is a circle 2m. The tank is 1m under the ground, the density of gasoline is $719 kg/m^3$ and let $g = 9.81 m/s^2$ be the gravitational constant.

Solution: First we think about how far we need to pump gasoline at least and at most and get that the minimal distance is from the top of the tank to the ground, 1m, and the maximal distance from the bottom of the tank to the ground: 5m which is composed as 4m, the height of the tank plus 1m, distance from the top of the tank to the ground. Hence is good to choose x to be the depth from the ground as our variable.

Cross-section area: Now we think about what a horizontal cross-section of our tank looks like. This is a rectangle with length constant 12m and width dependent on the depth x. The width follows the equation of a circle, where we have that: At a depth x going straight down from the ground to the center of the circle, the distance from the center of the circle to that point (denoted by a) and the distance from the point to the side wall (denoted by b) is described by the equation defining a circle: $a^2 + b^2 = r^2$, where here $r = 2$m. Now a is related to x by $a = 3 - x$. Hence we get that $b = \sqrt{4 - (3 - x)^2}$. So the entire width at depth x is twice this value, hence $w(x) = 2 \cdot \sqrt{4 - (3 - x)^2}$.

Hence the cross section area at depth x is $A(x) = 12 \cdot 2 \cdot \sqrt{4 - (3 - x)^2}$. Now we integrate over the bounds given by the minimal and maximal lifting distance. The gasoline at depth x of surface area $A(x)$ needs to be lifted x meters, hence we multiply $A(x)$ by x for the work:

$$W = \int_1^5 \rho \cdot g \cdot A(x) \cdot x \, dx.$$

Filling in, we obtain

$$W = \int_1^5 \rho \cdot g \cdot 12 \cdot 2 \cdot \sqrt{4 - (3 - x)^2} \cdot x \, dx.$$

First we substitute $u = (3 - x)$ to get the term under the root to a purely quadratic term:

$$W = \int_1^5 \rho \cdot g \cdot 12 \cdot 2 \cdot \sqrt{4 - u^2} \cdot (3 - u) \, dx.$$

Now we expand the integrand and separate the result into two integrals:

$$W = \int_1^5 \rho \cdot g \cdot 12 \cdot 2 \cdot \sqrt{4 - u^2} \cdot 3dx - \int_1^5 \rho \cdot g \cdot 12 \cdot 2 \cdot \sqrt{4 - u^2} \cdot u \, dx.$$

The first one is solved via a trigonometric substitution $u = \sin(t)$ and the second part by simply substituting $t = u^2$. As a final result we get

$$W = 3190878.462 \, J.$$