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Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Math	252/4	All
Examination	**Date**	**Pages**
Final	April 2011	2
Instructors		**Course Examiner**
E. Cohen, F. Thaine		F. Thaine

Special Instructions: ▷ Only approved calculators are allowed.

Answer ten questions. All questions have equal value.

1. Calculate the orthogonal projection of the vector $(1,1,2)$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $(2,0,1)$ and $(1,2,0)$, and find the distance from the point $(1,1,2)$ to the plane W.

2. Let $V = \mathbb{R}^3$, the inner product space with the standard inner product. By means of the Gram-Schmidt process, find an orthogonal basis of V by using the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$ as the starting basis .

3. Let $A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix}$. Let $T = L_A$; that is

   $T : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear operator given by $T(X) = AX$.

   a) Find a basis β of the T-cyclic subspace W of $\mathbb{R}^4$ generated by the vector

   $v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ and find $[T_W]_\beta$, where $T_W : W \to W$ is the restriction of T to W.

   b) Find the characteristic polynomial of T_W and determine if T_W is diagonalizable.

4. Find the general solution to the following system of linear differential equations.

$$\begin{aligned} x' &= x - z \\ y' &= 2x + 4y \\ z' &= 3z \end{aligned}$$

5. Use the least squares approximation to find the best fit with a linear function to the following data: $\{(0,2),(1,1),(2,3),(3,4)\}$. Compute the error of the best fit.

6. Let $S = \{(1,0,i),(1,2,1)\}$ in $\mathbb{C}^3$. Compute $S^{\perp}$.

7. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$. The characteristic polynomial of A is $-(t-2)^2(t+4)$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$ (that is $Q^tAQ = D$). Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

8. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Show that A is a normal matrix. Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$ (that is $U^*AU = D$). Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

9. Let $A = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}$. The characteristic polynomial of A is $-(t-2)^2(t-1)$. Find an invertible matrix Q and a Jordan matrix J such that $Q^{-1}AQ = J$. Find bases β_1 and β_2 for the generalized eigenspaces K_{λ_1} and K_{λ_2} of A, respectively, such that $\beta = \beta_1 \cup \beta_2$ is a Jordan basis for A.

10. If $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$ is defined by $T(A) = 2A + A^t$, for all $A \in V$. Find a Jordan canonical form J of T.

11. Let V be an inner product vector space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$, for all $x \in V$. First prove that T is linear, then show that T^* exists and find an explicit expression for it.

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	252	All
Examination	**Date**	**Pages**
Final	April 2010	2
Instructor		**Course Examiner**
E. Cohen, F. Thaine		E. Cohen

Special Instructions:

- ▷ No calculators are allowed.
- ▷ Answer 10 questions. All questions have equal value.
- ▷ Justify all your answers.

MARKS

1. Let $V = P_2(\mathbb{R})$, the vector space of real polynomials of degree less than or equal to 2. Define on V an inner product by $< f, g, >= \int_{-1}^{1} f(x)g(x)dx$. Find an orthogonal basis of V using $B = \{1, x, x^2\}$ as the starting basis.

2. Consider the inner product space $\mathbb{R}^4$ with standard inner product. Find a basis for the solution space S of the system:

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 0$$
$$x_1 + 5x_2 + 3x_3 + 3x_4 = 0,$$

and find a basis for $S^{\perp}$.

3. Let T be a linear operator on an inner product space V such that $< Tu, v >= - < u, Tv >$ for all $u, v, \in V$. Show if λ is an eigenvalue of T, than $\lambda + \lambda = 0$.

4. Let u be a unit vector in an n-dimensional inner product vector space V over $\mathbb{R}$. Extend u to an orthonormal basis $\beta = \{u, u_2, ...u_n\}$. Let $T : V \to V$ be defined by $Tx = x - 2 < x, u > u$, for all $x \in V$. Find $T\beta$, the image of the basis β of V. Show that T is an orthogonal operator and show that for any matrix A representing T, det $A = -1$.

5. (a) Let T be a normal linear operator on a finite-dimensional inner product space V.
Let $x \in N(T)$. Compare $\|Tx\|$ and $\|T^*x\|$. Show that $N(T) = N(T^*)$.

(b) Let A be a 3 x 3 matrix such that $A^2 = I$ and $A \neq I$ and $A \neq -I$; is A diagonalizable? Show that rank of one of $A + I$ and $A - I$ is 1 and the rank of the other is 2.

6. Let T be a linear operator on a complex inner product space V.

(a) Prove if $< Tu, v >= 0$ for all $u, v \in V$, then $T = T_\circ$, the zero linear transformation

(b) Let $< Tx, x >= 0$ for all $x \in V$. Expand the inner products $< T(x+y), x+y >$ and $< T(x + iy), x + iy >$. Show that $T = T_\circ$.

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (2x - y, 2y - 2z, 2z).$$

Find the basis of each generalized eigenspace of T and find the Jordan Canonical form J of T.

8. Let $A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix}$, its characteristic polynomial is $-(t-1)(t-2)^2$.

Find J the Jordan Canonical form of A and an invertible matrix Q such that $Q^{-1}AQ = J$.

9. Find all possible Jordan Canomical forms for the matrix whose characteristic polynomial is $c(t) = (t-2)^4(t-3)^3$ and whose minimal polynomial is $m(t) = (t-2)^2(t-3)$. Justify for your answer.

10. Let $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -3 & 2 \end{pmatrix}$.

Find an orthogonal matrix Q and a diagonal mattix D such that $Q^{-1}AQ = D$.

11. Let $T(x, y, z, u) = (2x + 2z, 2y + 6u, 2z, 2u)$. Find the Jordan canonical form J of T and a Jordan canonical basis β of $\mathbb{R}^4$.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	252	All
Examination	**Date**	**Pages**
Final	April 2009	2
Instructors		**Course Examiner**
D. Dryanov, F. Thaine		F. Thaine

Answer ten questions. All questions have equal value.

1. Calculate the orthogonal projection of the vector $(1, -1, 1)$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $(2, 2, 1)$ and $(1, 0, 1)$, and find the distance from the point $(1, -1, 1)$ to the plane W.

2. By using the Gram-Schmidt process, find an orthogonal basis of $\mathbb{R}^3$ containing the vector $(2, 2, 1)$.

3. Use the least squares approximation to find the best fit with a linear function to the following data: $\{(-1, 1), (0, 3), (1, 4), (2, 6)\}$. Compute the error of the best fit.

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The characteristic polynomial of A is $t^2(t-3)$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$ (that is $Q^tAQ = D$). Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

5. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Find an invertible matrix Q and a Jordan matrix J such that $Q^{-1}AQ = J$. Find bases β_1 and β_2 for the generalized eigenspaces K_{λ_1} and K_{λ_2} of A, respectively, such that $\beta = \beta_1 \cup \beta_2$ is a Jordan basis for A.

6. Find the general solution to the following system of linear differential equations.

$$\begin{aligned} x' &= 2x \\ y' &= 2x - 2y \\ z' &= \qquad\; y + 3z. \end{aligned}$$

7. Denote by $P_4(\mathbb{R})$ the vector space of all polynomials of degree ≤ 4 with real coefficients. Let $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be the linear operator defined by $T(f(x)) = f''(x)$ for each $f(x)$ in $P_4(\mathbb{R})$.

   a) Find a basis β of the T-cyclic subspace W of $P_4(\mathbb{R})$ generated by the vector x^4, and find $[T_W]_\beta$, where $T_W : W \to W$ is the restriction of T to W.

   b) Find the characteristic polynomial of T_W and determine if T_W is diagonalizable.

8. Let $A = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$. Show that A is a normal matrix. Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$ (that is $U^*AU = D$). Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

9. a) Let A be a square matrix and let $f(t)$ be its characteristic polynomial. Prove that A is invertible if and only if $f(0) \neq 0$. (Note: $f(0)$ is the constant term of $f(t)$.)

   b) Let A be a 3×3 matrix and I the 3×3 identity matrix. Show that the dimension of $\text{span}(\{I, A, A^2, A^3, ...\})$ is ≤ 3. (Hint: Apply the Cayley-Hamilton Theorem to the characteristic polynomial of A and show that we can write A^3, A^4, ... in terms of I, A and A^2.)

10. Let V be a finite dimensional inner product vector space over $\mathbb{C}$, and let T be a linear operator on V.

    a) Give the definitions of a self-adjoint operator on V and of a unitary operator on V.

    b) Show that if T is self-adjoint, then the eigenvalues of T are real.

    c) Show that if T is unitary, then the eigenvalues of T have absolute value 1.

11. Let V be a vector space, T a linear operator on V and λ an eigenvalue of T.

    a) Define the generalized eigenspace K_λ of T corresponding to λ.

    b) Prove that K_λ is a subspace of V.

    c) Prove that K_λ is a T-invariant subspace of V.

All questions have equal value.

1. Let A be an $n \times n$ matrix with entries in F. Suppose that A has two distinct eigenvalues λ_1 and λ_2 and that the dimension of the eigenspace E_{λ_1} is $n-1$. Prove that A is diagonalizable.

2. a) Let A be a 4×4 matrix with characteristic polynomial $f(t) = t^4 + 2t^2 - 3t + 5$. Find an expression of A^{-1} as a polynomial in A.

   b) Let T be a linear operator on a finite dimensional vector space V. Let λ be an eigenvalue of T. Prove that the eigenspace E_λ is a T-invariant subspace of V.

3. Let V be a finite dimensional vector space, T a linear operator on V, and W a T-invariant subspace of V. Let $\alpha = \{v_1, \ldots, v_r\}$ be a basis of W. Complete α to a basis $\beta = \{v_1, \ldots, v_r, v_{r+1} \ldots, v_n\}$ of V. Denote by T_W the restriction of T to W. Let $B = [T_W]_\alpha$ and $A = [T]_\beta$.

   a) Show that $A = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$. For some $r \times (n-r)$ matrix C and some $(n-r) \times (n-r)$ matrix D. (Here O denotes the $(n-r) \times r$ zero matrix.)

   b) Use part (a) to show that the characteristic polynomial of T_W divides the characteristic polynomial of T. (You can use any properties of determinants.)

4. a) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,1,1)$ and $(1,2,3,4)$. Find the orthogonal projection of the vector $(0,3,5,6)$ on W.

   b) For the following set of data use the least squares approximation to find the best fit with a linear function, and compute the error: $\{(1,0), (2,3), (3,5), (4,6)\}$.

5. Find an orthonormal basis of $\mathbb{R}^3$ containing the vector $(2/3, 1/3, 2/3)$.

6. Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$. The characteristic polynomial of A is $-t^2(t-6)$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

7. Let $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

   a) Find the singular values of B.
   b) Find the singular value decomposition of B. That is, find the matrices U (orthogonal), V (orthogonal) and Σ in the decomposition $B = U\Sigma V^t$. (You can use the results from question 6.)

8. Let $A = \begin{bmatrix} 3 & -1 & -4 \\ 3 & -1 & -3 \\ 1 & -1 & -2 \end{bmatrix}$. The characteristic polynomial of A is $-(t+1)^2(t-2)$. Find an invertible matrix Q and a Jordan matrix J such that $Q^{-1}AQ = J$. Find bases β_1 and β_2 for the generalized eigenspaces K_{λ_1} and K_{λ_2} of A, respectively, such that $\beta = \beta_1 \cup \beta_2$ is a Jordan basis for A.

Solve ONLY two of the following three questions.

9. Find the general solution to the following system of linear differential equations.

$$\begin{aligned} x' &= x - y \\ y' &= \quad 2y + z \\ z' &= \qquad 3z. \end{aligned}$$

10. Let $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$. Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$. Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

11. a) If V is an inner product space and T is a linear operator on V, define the adjoint operator T^*.

    b) Show that if T is self-adjoint, then the eigenvalues of T are real.