MATH 252 Linear Algebra II: Review

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This text is a brief survey of what is needed for exams of MATH 252. Basic definitions are given, along with main theorems and some results. Most importantly, however, this text contains some of the types of questions that a student should expect to see. Everything is taken from the textbook, past midterms, homework assignments, and final exams. There is no guarantee that these questions are representative of the exams *this* semester or that you will see these questions *exactly*. Take this review package as being *extra* help with your studying, not your sole source.

The following represents the notational conventions that will used throughout this text. They may be different from those used by your professor; the trick is to *understand* what they mean instead of *memorizing* what they mean.

- For a field, the symbol $\mathbb{F}$ will be used. Possible specific fields include $\mathbb{C}$, the set of all complex numbers; $\mathbb{R}$, the set of all real numbers; or $\mathbb{Q}$, the set of all rational numbers.
- The vector space of n-tuples of elements from a given field $\mathbb{F}$ is denoted $\mathbb{F}^n$.
- The vector space of all polynomials of degree less than or equal to n with coefficients from a field $\mathbb{F}$ is denoted $\mathcal{P}_n(\mathbb{F})$. For the space of all polynomials, the notation $\mathcal{P}(\mathbb{F})$ is used.
- The vector space of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is denoted $\mathcal{M}_{m,n}(\mathbb{F})$. For square matrices (i.e. those for which $m = n$), the notation $\mathcal{M}_n(\mathbb{F})$ is used.
- The vector space of all functions from a non-empty set S to a field $\mathbb{F}$ is denoted $\mathcal{F}(S, \mathbb{F})$. For real-valued functions of a real variable, the notation $\mathcal{F}(\mathbb{R})$ is used. Occasionally, we may consider the vector space of all continuous real-valued functions defined on an interval $[a, b]$; this is denoted $\mathcal{C}([a, b])$.

Note that for the purpose of MATH 252, students may generally interpret the arbitrary field $\mathbb{F}$ as the familiar field $\mathbb{C}$, hence the vector space $\mathbb{F}^n$ as the vector space $\mathbb{C}^n$.

Now that you have spent an entire semester acquainting yourself with the fundamental language of linear algebra you should be comfortable with one of the most important themes of MATH 251, the fact that matrices and linear transformations are the same under the microscope of linear algebra. This course will attempt to cover two new broad themes: the first is the rich theory one acquires when vector spaces are endowed with additional structures, and the second is the answer to the question of where to go next if a matrix is not diagonalizable. Still, though, most of the theory developed concerns vector spaces that are finite dimensional; in fact, this is what gives us the correspondence between matrices and linear transformations. The next step for students of mathematics is to consider functional analysis, the branch of analysis that concerns itself primary with linear algebra at the infinite-dimensional level. It is much more abstract than what you are used to but has many beautiful results and has many topics that are comparable to what you are learning now.

I hope you enjoy linear algebra as much as I do.

Section 1: Diagonalization & Invariance

Definitions

Let V be a vector space and let $T \in \mathcal{L}(V)$.

1. A scalar λ is called an eigenvalue of T if $T(v) = \lambda v$ for some non-zero vector $v \in V$ called an eigenvector corresponding to λ.
2. For V finite dimensional, the characteristic polynomial of T is defined to be $f(t) = \det(T - tI_V)$.
3. A polynomial $f(t)$ splits over a field $\mathbb{F}$ if there exist scalars $c, a_0, a_1, \cdots, a_n \in \mathbb{F}$ such that
$$f(t) = c(t - a_0)(t - a_1) \cdots (t - a_n),$$
i.e. $f(t)$ can be factored into the product of linear factors with respect to the field $\mathbb{F}$.
4. The algebraic multiplicity of an eigenvalue is the largest positive integer k for which $(t-\lambda)^k$ is a factor of the characteristic polynomial of T.
5. The eigenspace of T corresponding to the eigenvalue λ is defined as $E_\lambda = \{v \in V : T(v) = \lambda v\} = N(T - \lambda I_V)$.
6. For V finite dimensional, the linear operator T is called diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.
7. A subspace W of V is said to be T-invariant if $T(W) \subseteq W$, i.e. $T(w) \in W$ for all $w \in W$.
8. The restriction of T to a T-invariant subspace W is the function $T_W : W \to W$ defined by $T_W(w) = T(w)$ for all $w \in W$.
9. For a non-zero vector $v \in V$, the subspace $W = \text{span}\{v, T(v), T^2(v), \cdots\}$ is called the T-cyclic subspace of V generated by v.

Theorems

Let V be a finite-dimensional vector space and let $T \in \mathcal{L}(V)$.

1. Let $f(t)$ be the characteristic polynomial of T. Then, $\lambda \in \mathbb{C}$ is an eigenvalue for T if and only if $f(\lambda) = 0$.
2. T is diagonalizable if and only if there exits an ordered basis for V consisting an eigenvectors of T.
3. T is diagonalizable if and only if the following hold: its characterisitic polynomial splits, and the multiplicity of each eigenvalue λ is equal to $\dim(V) - \mathrm{rank}(\mathrm{T} - \lambda \mathrm{I})$.
4. Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, and that it is the coefficient matrix to a system of differential equations, i.e. $x'(t) = Ax(t)$ for a differentiable function $x : \mathbb{R} \to \mathbb{R}^n$. Then, $\tilde{x}$ is a solution to the system if and only if it is in the form
$$\tilde{x}(t) = \sum_{i=1}^{k} z_i e^{\lambda_i t}$$
where $z_i \in E_{\lambda_i}$.
5. If W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.

6. If W is a T-cyclic subspace of V generated by $v \in V$ such that $\dim(W) = k$, then the set $\beta = \{v, T(v), T^2(v), \cdots, T^{k-1}(v)\}$ is a basis for W. Moreover, the following gives us a technique for computing the characteristic polynomial of T_W, $f(t)$, without resorting to determinants:

$$[T(v)]_\beta = \begin{pmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{k-1} \end{pmatrix} \Rightarrow f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

7. (Cayley-Hamilton) If $f(t)$ is the characteristic polynomial of T, then $f(T) = T_0$, the zero transformation.

Problems

1. Find the general solution to the following systems of differential equations:
   (a)
   $$\begin{aligned} x_1' &= x_1 - x_2 + x_3 \\ x_2' &= 2x_2 - x_3 \\ x_3' &= 3x_3 \end{aligned}$$
   (b)
   $$\begin{aligned} x' &= x + y \\ y' &= 2x - y \end{aligned}$$
2. For each linear operator T on the vector space V find the T-cyclic subspace W generated by the vector v, compute the characteristic polynomial of T_W in two ways and verify that it divides the characteristic polynomial of T:
   (a) $V = \mathbb{R}^4$, $T(a, b, c, d) = (a + b, b - c, a + c, a + d)$, $v = e_1$.
   (b) $V = \mathcal{M}_2$, $T(A) = A^t$, $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
3. For each linear operator T on the vector space V, determine whether the subspace W is T-invariant.
   (a) $V = \mathbb{R}^3$, $T(a, b, c) = (a + b + c, a + b + c, a + b + c)$, $W = \{(t, t, t) : t \in \mathbb{R}\}$.
   (b) $V = \mathcal{C}([0, 1])$, $T\big(f(t)\big) = \left[\int_0^1 f(x)\mathrm{d}x\right] t$, $W = \{f \in \mathcal{C}([0, 1]) : f(t) = at + b\}$.

Solutions

1. (a) Writing the system of differential equations in matrix form, we have that

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Denoting the matrix of coefficients by A, we may find its characteristic polynomial:

$$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & -1 & 2 \\ 0 & 2-t & -1 \\ 0 & 0 & 3-t \end{vmatrix} = (1-t)(2-t)(3-t)$$

so that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Therefore, the general solution to the system of differential equations is $x(t) = z_1 e^t + z_2 e^{2t} + z_3 e^{3t}$ where each $z_i \in E_{\lambda_i}$ for $i = 1, 2, 3$.

(b) Writing the system of differential equations in matrix form, we have that

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Denoting the matrix of coefficients by A, we may find its characteristic polynomial:

$$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 \\ 3 & -1-t \end{vmatrix} = (1-t)(-1-t) - 3 = t^2 - 4 = (t-4)(t+4)$$

so that the eigenvalues of A are $\lambda_1 = 2, \lambda_2 = -2$. Therefore, the general solution to the system of differential equations is $x(t) = z_1 e^{2t} + z_2 e^{-2t}$ where each $z_i \in E_{\lambda_i}$ for $i = 1, 2, 3$.

We can easily find the eigenspaces:

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so that we may take $z_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; and

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so that we may take $z_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

This gives us

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-2t}.$$

2. (a) So we begin by finding the T-cyclic subspace generated by v:

$$W = \operatorname{span}\{v, T(v), T^2(v), \cdots\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} \right\}.$$

We know that this is where to stop in our span because $T^3(v) = \begin{pmatrix} 0 \\ -3 \\ 3 \\ 3 \end{pmatrix} = -3T(v) + 3T^2(v)$. Since we can write this as $0 = 0v + 3T(v) - 3T^2(v) + T^3(v)$, we have that $f_{T_W}(t) = (-1)^3(3t - 3t^2 + t^3) = -3t + 3t^2 - t^3$. Using determinants, we have that the matrix representing T_W with respect to the basis found for above for W is

$$[T_W] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

since $T(v) = 0v + 1T(v) + 0T^2(v)$, $T(T(v)) = 0v + 0T(v) + 1T^2(v)$, and, as before, $T(T^2(v)) = 0v - 3T(v) + 3T^2(v)$. So,

$$f_{T_W}(t) = \begin{vmatrix} -t & 0 & 0 \\ 1 & -t & -3 \\ 0 & 1 & 3-t \end{vmatrix} = -t[(-t)(3-t) + 3] = -t[t^2 - 3t + 3] = -t^3 + 3t^2 - 3t,$$

which is what we found before.

To find the characteristic polynomial of T, we begin with the standard basis σ and write

$$[T]_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

so that

$$f_T(t) = \begin{vmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & -1 & 0 \\ 1 & 0 & 1-t & 0 \\ 1 & 0 & 0 & 1-t \end{vmatrix} = t^4 - 4t^3 + 6t^2 - 3t.$$

It's not pretty, but one may see that $f_{T_W}(t)$ divides $f_T(t)$ by long division.

(b) So we begin by finding the T-cyclic subspace generated by v:

$$W = \text{span}\{v, T(v), T^2(v), \cdots\} = \text{span}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}.$$

We know that this is where to stop in our span because $T(v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = v$. Since we can write this as $0 = -v + T(v)$, we have that $f_{T_W}(t) = (-1)^1(-1+t) = 1-t$. Using determinants, we have that the matrix representing T_W with respect to the basis found for above for W is

$$[T_W] = \begin{pmatrix} 1 \end{pmatrix}$$

since $T(v) = v$. So,

$$f_{T_W}(t) = \begin{vmatrix} 1-t \end{vmatrix} = 1-t,$$

which is what we found before.

To find the characteristic polynomial of T, we begin with the standard basis σ and write

$$[T]_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that

$$f_T(t) = \begin{vmatrix} 1-t & 0 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 1-t \end{vmatrix} = (1-t)^2(t^2-1) = -(1-t)^3(t+1).$$

It's easy to see that $f_{T_W}(t)$ divides $f_T(t)$.

3. (a) We take an element of W, i.e. (t,t,t) for some $t \in \mathbb{R}$. Then, $T(t,t,t) = (3t,3t,3t) \in W$. Therefore, W is T-invariant.

   (b) We take an element of W, i.e. $f(t) = at + b$ for some $a, b \in \mathbb{R}$. Then, $T\big(f(t)\big) = \left[a\frac{t^2}{2} + bt\right]_0^1 t = [\frac{a}{2} + b]t \in W$. Therefore, W is T-invariant.

Section 2: Inner Product Spaces

Definitions

Let V be a vector space over a field $\mathbb{F}$.

1. We say that V is an inner product space if V is endowed with an inner product, a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that the following hold for all $u, v, w \in V$ and $c \in \mathbb{F}$:
    (a) $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$,
    (b) $\langle cu, v \rangle = c\langle u, v \rangle$,
    (c) $\overline{\langle u, v \rangle} = \langle v, u \rangle$,
    (d) $\langle u, u \rangle > 0$ if $u \neq \theta$.
2. We say that V is an normed space if V is endowed with a norm, a mapping $\|\cdot\| : V \to \mathbb{R}$ such that the following hold for all $u, v \in V$ and $c \in \mathbb{F}$:
    (a) $\|cu\| = |c|\|u\|$,
    (b) $\|u + v\| \leq \|u\| + \|v\|$,
    (c) $\|u\| \geq 0$ and $\|u\| = 0 \Leftrightarrow x = \theta$.
3. If V is an inner product space, we say that a subset S is orthogonal if for each distinct $u, v \in S$, $\langle u, v \rangle = 0$. We say that a subet S is orthonormal if it is orthogonal and for each $u \in S$, $\|u\| = 1$. If S is non-empty, its orthogonal complement is defined to be $S^\perp = \{u \in V : \langle u, v \rangle = 0 \text{ for all } v \in S\}$. An orthonormal basis for V is a basis that is also an orthonormal set.

Theorems

Let V be an inner product space over a field $\mathbb{F}$.

1. For $u, v, w \in V$ and $c \in \mathbb{F}$, the following hold:
    (a) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
    (b) $\langle u, cv \rangle = \bar{c}\langle u, v \rangle$,
    (c) $\langle u, \theta \rangle = \langle \theta, u \rangle = 0$,
    (d) $\langle u, u \rangle = 0$ if and only if $u = \theta$,
    (e) if $\langle u, v \rangle = \langle u, w \rangle$ for all $u \in V$, then $v = w$,
    (f) $|\langle u, v \rangle| \leq \|u\|\|v\|$.
2. Via the relation $\|u\| = \sqrt{\langle u, u \rangle}$, V is also a normed space.
3. Let $S = \{v_1, v_2, \cdots v_k\}$ be an orthonormal subset consisting of non-zero vectors. If $u \in \operatorname{span}(S)$, then
$$u = \sum_{i=1}^{k} \langle u, v_i \rangle v_i.$$
4. (Gram-Schmidt) Let $S = \{w_1, w_2, \cdots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \cdots, v_n\}$ where $v_1 = w_1$ and
$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\langle v_j, v_j \rangle} v_j, \qquad \text{for } 2 \leq k \leq n.$$
Then S' is an orthogonal set of non-zero vectors such that $\operatorname{span}(S') = \operatorname{span}(S)$.

5. If V is finite dimensional and not the zero vector space, then V has an orthonormal basis β; moreover, if $\beta = \{v_1, v_2, \cdots, v_n\}$ and $u \in V$, then
$$u = \sum_{i=1}^{k} \langle u, v_i \rangle v_i.$$

6. (The Projection Theorem) Let W be a finite-dimensional subspace of V and let $y \in V$. Then, there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that u is the closest vector in W to y. z is the closest vector in $W^{\perp}$ to y, and $y = u + z$.

7. If $\dim(V) < \infty$ and W is a subspace of V, then $\dim(V) = \dim(W) + \dim(W^{\perp})$.

8. (The Riesz Representation Theorem) Let $\dim(V) < \infty$ and $g : V \to \mathbb{F}$ be a linear transformation. Then, there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

9. Let $\dim(V) < \infty$ and let $T \in \mathcal{L}(V)$. Then, there exists a unique linear transformation $T^* : V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$.

10. If $T, U \in \mathcal{L}(V)$ and $c \in \mathbb{F}$, then
    (a) $(T + U)^* = T^* + U^*$
    (b) $(cT)^* = \overline{c}T^*$
    (c) $(TU)^* = U^*T^*$
    (d) $T^{**} = T$
    (e) $I^* = I$.

Problems

1. Identify why the following are not inner products on the indicated vector space:
    (a) $\langle v_1, v_2 \rangle = \det(v_1|v_2)$ on $\mathbb{R}^2$;
    (b) $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $\mathcal{M}_2(\mathbb{R})$
    (c) $\langle f, g \rangle = \int_0^1 f'(t)g(t)\mathrm{d}t$ on $\mathcal{P}_2(\mathbb{R})$.

2. Consider $\mathcal{C}[0,1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\mathrm{d}t$. Verify the Cauchy-Schwartz and triangle inequalities for $f(t) = t$ and $g(t) = e^t$.

3. Apply Gram-Schmidt on the subset $S = \{\sin(t), \cos(t), 1, t\}$ to find an orthogonal basis for $\mathrm{span}(S)$ with inner product $\langle f, g \rangle = \int_0^{\pi} f(t)g(t)\mathrm{d}t$.

Solutions

1. (a) This inner product violates the property that $\langle v, v \rangle > 0$ for $v \neq \theta$. For example, $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0.$
    (b) This inner product violates linearity in the first start. For example, $\langle 2I, I \rangle = \mathrm{tr}(3I) = 6 \neq 8 = 2\mathrm{tr}(2I) = 2\langle I, I \rangle$.
    (c) This inner product violates the property that $\langle v, v \rangle > 0$ for $v \neq \theta$. For example, $\langle 1, 1 \rangle = \int_0^1 0\mathrm{d}t = 0$.

2. We have that

$$\langle t, e^t \rangle = \int_0^1 te^t \mathrm{d}t = e^t(t-1)|_0^1 = 1,$$

$$\langle t, t \rangle = \int_0^1 t^2 \mathrm{d}t = t^3/3|_0^1 = 1/3,$$

$$\langle e^t, e^t \rangle = \int_0^1 e^{2t} \mathrm{d}t = 1/2e^{2t}|_0^1 = 1/2(e^2-1).$$

So we may check that

$$\|t+e^t\| = \sqrt{7/3+1/2(e^2-1)} \leq \sqrt{1/3} + \sqrt{1/2(e^2-1)} = \|t\| + \|e^t\|$$

and

$$|\langle t, e^t \rangle| = 1 \leq \sqrt{1/6(e^2-1)} = \|t\|\|e^t\|.$$

3. We begin by defining $v_1 = \sin(t)$ and $W_1 = \mathrm{span}\{\sin(t)\}$. Then,

$$v_2 = \cos(t) - \mathrm{proj_{W_1}cos(t)} = \mathrm{cos(t)} - \frac{\langle \mathrm{cos(t), sin(t)} \rangle}{\langle \mathrm{sin(t), sin(t)} \rangle} \mathrm{sin(t)}.$$

We fine these inner products by evaluating the integrals:

$$\langle \cos(t), \sin(t) \rangle = \int_0^\pi \cos(t)\sin(t)\mathrm{d}t = \int_0^\pi 1/2\sin(2t)\mathrm{d}t = -1/4\cos(2t)|_0^\pi = 0,$$

$$\langle \sin(t), \sin(t) \rangle = \int_0^\pi \sin^2(t)\mathrm{d}t = \int_0^\pi 1/2(1-\cos(2t))\mathrm{d}t = 1/2(t-1/2\sin(2t))|_0^\pi = \pi/2$$

so that

$$v_2 = \cos(t)$$

and $W_2 = \mathrm{span}\{\,\sin(t), \cos(t)\}$. Next,

$$v_3 = 1 - \mathrm{proj_{W_2}}1 = 1 - \frac{\langle 1, \mathrm{sin(t)} \rangle}{\langle \mathrm{sin(t), sin(t)} \rangle} \mathrm{sin(t)} - \frac{\langle 1, \mathrm{cos(t)} \rangle}{\langle \mathrm{cos(t), cos(t)} \rangle} \mathrm{cos(t)}.$$

So we need to calculate even more inner products.

$$\langle 1, \sin(t) \rangle = \int_0^\pi \sin(t)\mathrm{d}t = 2,$$

$$\langle 1, \cos(t) \rangle = \int_0^\pi \cos(t)\mathrm{d}t = 0,$$

$$\langle \cos(t), \cos(t) \rangle = \int_0^\pi \cos^2(t)\mathrm{d}t = \pi/2$$

so that

$$v_3 = 1 - 4/\pi\sin(t)$$

and $W_3 = \mathrm{span}\{\,\sin(t), \cos(t), 1-4/\pi\sin(t)\}$. Lastly,

$$v_4 = t - \frac{\langle t, \sin(t) \rangle}{\langle \sin(t), \sin(t) \rangle}\sin(t) - \frac{\langle t, \cos(t) \rangle}{\langle \cos(t), \cos(t) \rangle}\cos(t) - \frac{\langle t, 1-\frac{4}{\pi}\sin(t) \rangle}{\langle 1-\frac{4}{\pi}\sin(t), 1-\frac{4}{\pi}\sin(t) \rangle}(1-\frac{4}{\pi}\sin(t)).$$

We need to tediously calculate more integrals to finally get the last vector.

$$\langle t, \sin(t) \rangle = \int_0^\pi t\sin(t)\mathrm{d}t = -t\cos(t) + \sin(t)|_0^\pi = \pi,$$

$$\begin{aligned}
\langle t, \cos(t) \rangle &= \int_0^\pi t\cos(t)\mathrm{d}t = t\sin(t) - \cos(t)|_0^\pi = 2, \\
\langle t, 1 - \frac{4}{\pi}\sin(t) \rangle &= \int_0^\pi (t - \frac{4}{\pi}t\sin(t))\mathrm{d}t = \frac{\pi^2}{2} - 4, \\
\langle 1 - \frac{4}{\pi}\sin(t), 1 - \frac{4}{\pi}\sin(t) \rangle &= \int_0^\pi (1 - \frac{4}{\pi}\sin(t))^2\mathrm{d}t = \pi - \frac{8}{\pi}
\end{aligned}$$

so that

$$v_4 = t - 2\sin(t) - \frac{4}{\pi}\cos(t) - \frac{\pi}{2}(1 - \frac{4}{\pi}\sin(t)) = t - \frac{4}{\pi}\cos(t) - \frac{\pi}{2}.$$

Therefore, the orthogonal basis is

$$\beta = \left\{ \sin(t), \cos(t), 1 - \frac{4}{\pi}\sin(t), t - \frac{4}{\pi}\cos(t) - \frac{\pi}{2} \right\}.$$

Section 3: Operators

Definitions

Let V be a vector space over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V)$.

1. T is called normal if $T^*T = TT^*$.
2. T is called self-adjoint if $T = T^*$.
3. T is called unitary if $TT^* = T^*T = I$.

Theorems

Let V be a vector space over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V)$.

1. If T is normal, then the following are true:
   (a) $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$;
   (b) $T - cI$ is normal for every $c \in \mathbb{F}$;
   (c) if x is an eigenvector of T such that $T(x) = \lambda(x)$ then x is an eigenvector of T^* such that $T^*(x) = \overline{\lambda}x$;
   (d) if λ_1, λ_2 are distinct eigenvalues with corresponding eigenvectors x_1, x_2 then x_1 and x_2 are orthogonal.
2. Let $\dim(V) < \infty$ and $\mathbb{F} = \mathbb{C}$. Then, T is normal if and only if there exists an orthonormal basis for V of eigenvectors of T.
3. If T is self-adjoint and $\dim(V) < \infty$, then every eigenvalue of T is real.
4. Let $\dim(V) < \infty$ and $\mathbb{F} = \mathbb{R}$. Then, T is self-adjoint if and only if there exists an orthonormal basis for V of eigenvectors of T.
5. If $\dim(V) < \infty$, then the following are equivalent to T being unitary:
   (a) $\|T(x)\| = \|x\|$ for all $x \in V$;
   (b) $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$;
   (c) if β is an orthonormal basis for V, then $T(\beta)$ is also an orthonormal basis for V;
   (d) there exists an orthonormal basis β for V such that $T(\beta)$ is also an orthonormal basis for V;
   (e) $TT^* = T^*T = I$.

Section 4: Proofs

Problems

1. Let V be a vector space with subspace W and let $T \in \mathcal{L}(V)$. Show that if W is a T-invariant subspace, then $T_W \in \mathcal{L}(W)$.
2. Let V be a finite-dimensional vector space and let $T \in \mathcal{L}(V)$. Prove that the following subspaces of V are T invariant:
   (a) $\{\theta\}$ and V,
   (b) $N(T)$ and $R(T)$,
   (c) E_λ for any eigenvalue λ of T,
   (d) W, a T-cyclic subspace generated by a non-zero vector v.
3. Let V be an inner product space and S be a non-empty subset of V. Show that $S^\perp$ is a subspace of V.
4. Let V be an inner product over $\mathbb{R}$. Then, show that the polarization identity $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ holds for all $x, y \in V$.
5. Show that if $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner products on a vector space V that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$ is also an inner product on V.
6. Prove that the adjoint of a linear operator on a vector space is linear and unique.
7. Let $T \in \mathcal{L}(V)$ for a vector space V over $\mathbb{R}$. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. [Hint: Use the polarization identity.]
8. Let V be an inner product space, let $T \in \mathcal{L}(V)$, and let W is a T-invariant subspace of V. Prove the following:
   (a) $W^\perp$ is a T^*-invariant subspace of V,
   (b) if T is self-adjoint, then T_W is self-adjoint.
9. Let V be an inner product space and let $T \in \mathcal{L}(V)$. Show that the operators $\tilde{T} = \frac{1}{2}(T+T^*)$ and $\hat{T} = \frac{1}{2i}(T - T^*)$ are self-adjoint.
10. Let V be the vector space of all complex-valued continuous functions on the interval $[0,1]$ with the inner product
$$\langle f, g\rangle = \int\limits_0^1 f(t)\overline{g(t)}\,\mathrm{d}t.$$
Let $h \in V$ and define $T : V \to v$ to be $T(f) = hf$. Prove that T is unitary if and only if $|h(t)| = 1$ for $0 \leq t \leq 1$.

Do not forget to use your time wisely for studying and for writing your exam. Remember that mathematics is about two things: breathing and thinking. Study well, breathe deeply, and think as much as you need without rashly computing things.

Good luck!