

**FACULTY OF ENGINEERING AND COMPUTER SCIENCES  
CONCORDIA UNIVERSITY  
APPLIED ADVANCED CALCULUS (ENGR 233) FINAL EXAMINATION  
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**Special instructions: Do all problems  
No calculators are allowed**

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**PROBLEM No. 1 (10 MARKS)**

- (a) Find the equation of the plane which contains the three points  $(2, 1, 0)$ ,  $(3, 4, 0)$ , and  $(1, 1, 1)$
- (b) Find the parametric equations of the line through  $(3, 4, 0)$ , which is perpendicular to the plane found in part (a).

**PROBLEM No. 2 (10 MARKS)**

- (a) Let  $w = \sqrt{x^3 + y} + e^{xz}$ , and let  $x = 2t$ ,  $y = t^2$ , and  $z = t^{-1}$ . Use the multivariate chain rule to find  $\frac{dw}{dt}$  when  $t = 1$ .
- (b) Suppose that the temperature distribution of the plane is given by  $T(x,y) = 5x^2 + e^{xy}$ . If an ant is sitting at the point  $(2,3)$ , in which unit direction should it move in order to cool off as fast as possible? What is the directional derivative of  $T$  in that direction?

**PROBLEM No. 3 (10 MARKS).** Find point(s) on the surface  $z = 10 - x^2 - y^2$  at which the tangent plane is parallel to the plane  $x + \frac{3}{2}y + \frac{1}{2}z + d = 0$ , where  $d$  is a constant.

**PROBLEM No. 4 (10 MARKS).** Find the moment of inertia about the  $y$ -axis of the lamina that has the given shape and density.

$$\text{bounded by } x = 0, y = x, y = 1; \rho(x, y) = \sqrt{1 + y^4}$$

**PROBLEM No. 5 (10 MARKS).** For the scalar function

$$f(x, y, z) = e^{x^2} \cos z + z^4 \sin y$$

compute the following quantities if they **make sense**. If not, explain why.

- (a) **grad f**   (b) **div(grad f)**   (c) **grad(div f)**   (d) **curl(grad f)**   (e) **grad(grad f)**

**PROBLEM No. 6 (10 MARKS).** Compute the line integral

$$\int_C -y \, dx + x \, dy$$

for the following curves

- (a)  $C$  is the curve from  $(3,0)$  to  $(-3,0)$  lying along the ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$  in the  $xy$  plane,  $y \geq 0$ .
- (b)  $C$  is the closed curve in the  $xy$  plane described, in the counterclockwise sense, by  $r = 2 \cos \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

**PROBLEM No. 7 (10 MARKS).** Prove that the line integral

$$I = \int_C (1 + e^{-y}) \, dx - (xe^{-y} + 4y) \, dy$$

is independent of the path, and then evaluate  $I$  when  $C$  is any path from  $(1, 0)$  to  $(2, 1)$ .

**PROBLEM No. 8 (10 MARKS).** Find the outward flux of the radial vector field

$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  through the boundary of domain in  $\mathbb{R}^3$  given by two inequalities  $x^2 + y^2 + z^2 \leq 2$  and  $z \geq x^2 + y^2$ .

**PROBLEM No. 9 (10 MARKS).**  $D$  is the region of three-dimensional space that is given by the inequalities  $x^2 + y^2 + z^2 \leq 1$ ,  $4z^2 \leq x^2 + y^2 + z^2$  and  $z \geq 0$ . Find the volume of  $D$ .

**PROBLEM No. 10 (10 MARKS).** A field of force:  $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$  exists within a region of space.

- (a) Solve  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where the closed contour “ $C$ ” is the intersection of  $x^2 + y^2 = 1$  and  $z = y^2$  using Stokes Theorem. Show all of your work and justify your answer.
- (b) If the object were moved along the closed path “ $C$ ” 16 times, how much more work would be done than if the object were moved along the closed path just once? Justify your answer.

$$\cos \theta = \vec{a} \cdot \vec{b} / (\|\vec{a}\| \|\vec{b}\|)$$

$$\text{comp}_{\vec{b}} \vec{a} = \|\vec{a}\| \cos \theta = \vec{a} \cdot \hat{\vec{b}}$$

$$\text{proj}_{\vec{b}} \vec{a} = (\vec{a} \cdot \hat{\vec{b}}) \hat{\vec{b}}$$

$$\text{Area of a parallelogram} = \|\vec{a} \times \vec{b}\|$$

$$\text{Volume of a parallelepiped} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Equation of a line:

$$\vec{r} = \vec{r}_2 + t(\vec{r}_2 - \vec{r}_1) = \vec{r}_2 + t\vec{a}$$

Equation of a plane:  $ax + by + cz + d = 0$

$$\text{also: } [(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)] \cdot (\vec{r} - \vec{r}_1) = 0$$

$$\frac{d\vec{r}(s)}{ds} = \frac{d\vec{r}}{ds} \frac{ds}{dt}$$

$$\text{Length of a curve: } s = \int_{t_1}^{t_2} |\vec{r}'(t)| dt$$

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\| = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{a}(t) = \kappa v^2 \hat{\vec{N}} + \frac{dv}{dt} \hat{\vec{T}} = a_N \hat{\vec{N}} + a_T \hat{\vec{T}}$$

$$\hat{\vec{N}} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

$$\hat{\vec{T}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

The Binormal

$$\hat{\vec{B}} = \hat{\vec{T}} \times \hat{\vec{N}}$$

$$a_T = \frac{dv}{dt} = \frac{\|\vec{v} \cdot \vec{a}\|}{\|\vec{v}\|} \quad \& \quad a_N = \kappa v^2 = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|}$$

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad \& \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y}$$

$$D_u(F) = \nabla F \cdot \hat{u}, \quad \hat{u} = \text{unit vector}$$

Equation of Tangent Plane:

$$\vec{n}_o \cdot (\vec{r} - \vec{r}_o) = 0, \quad \vec{n}_o = \nabla F \text{ at P}$$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

equation of normal line to a surface:  $\vec{n}_o \times (\vec{r} - \vec{r}_o) = 0, \quad \vec{n}_o = \nabla F \text{ at P}$

$$\int_C F(x, y) ds = \int_a^b F(f(t), g(t)) \sqrt{[f']^2 + [g']^2} dt = \int_a^b F(x, f(x)) \sqrt{1 + [f']^2} dx$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n} dS$$

$$\iiint_S (\vec{F} \cdot \hat{n}) dS = \iiint_D (\text{div } \vec{F}) dV$$

$$\oint_C [P dx + Q dy] = \iint_R \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

$$\tilde{x} = \frac{\iiint_D x \rho(x, y, z) dV}{m}$$

$$m = \iiint_D \rho(x, y, z) dV$$

$$I_x = \iiint_D (y^2 + z^2) \rho(x, y, z) dV;$$

$$x = r \cos \theta, \quad y = r \sin \theta; \quad z = z; \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

$$dV = r dr d\theta dz$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}(y/x), \quad \phi = \tan^{-1}(\sqrt{x^2 + y^2}/z)$$

$$dV = \rho^2 \sin \phi d\rho d\phi d\theta$$