

CONCORDIA UNIVERSITY  
Faculty of Engineering and Computer Science  
Department of Engineering

FINAL EXAMINATION ENGR 213, 13. 4. 2007

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**Answer each question! Calculators Permitted!**

(10)#1...Solve the following two ordinary differential equations:

$$(A) \frac{dy}{dx} = \frac{y^2 e^x}{1 + e^{2x}}, \quad (B) \frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x \text{ with } y(\pi) = 4.$$

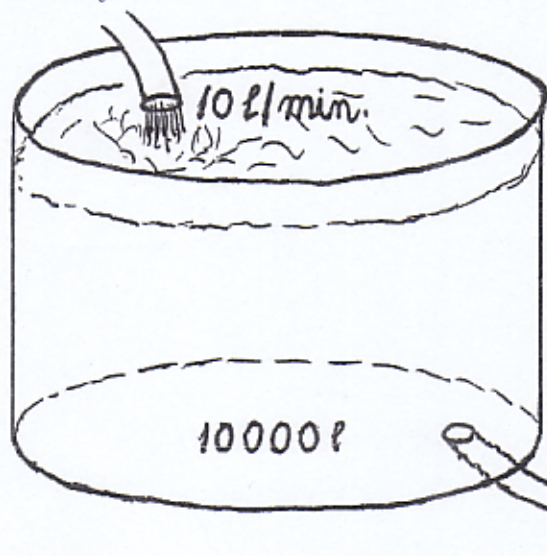
(10)#2...Solve the following ordinary differential equations by means of a suitable integrating factor  $\mu = \mu(x)$  or  $\mu = \mu(y)$ , which shall make it exact:

$$6xy dx + (4y + 9x^2)dy = 0. \text{ Note: } \frac{\mu'(x)}{\mu(x)} = \frac{M_y - N_x}{N} \text{ or } \frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M}.$$

(10)#3...Solve the following two ordinary differential equations by means of a suitable substitution:

$$(A) \frac{dy}{dx} = \frac{x + 3y}{3x + y}, \quad (B) \frac{dy}{dx} + y = 8xy^4.$$

(10)#4...A cylindrical tank is filled with 10000 liters of brine containing 500 kg of



salt initially. Entering from the top of the tank, at a rate of 10 l/min, is a saline solution containing 0.3 kg/l. From the well-stirred tank 10 l/min is discharged at the bottom.

- (A) How much salt is contained in the tank at time  $t$  (in minutes)?  
 (B) What does this tell you about the amount of salt in the tank as  $t \rightarrow \infty$ ?

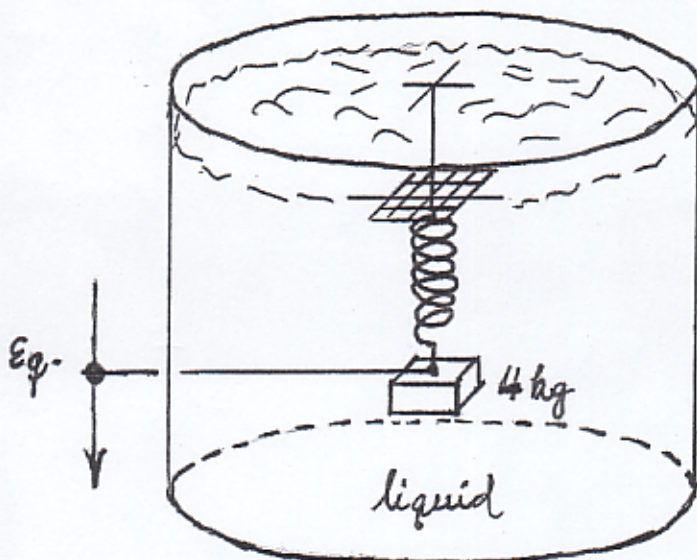
(10)#5... Solve the following linear homogeneous second order ordinary differential equations with the given initial conditions:

$$x^2 y'' - 6xy' - 18y = 0 \quad \text{satisfying: } y(1) = 2 \text{ and } y'(1) = 16.$$

(10)#6... Find the general solutions of the following linear inhomogeneous second order ordinary differential equation by means of variation of parameter:

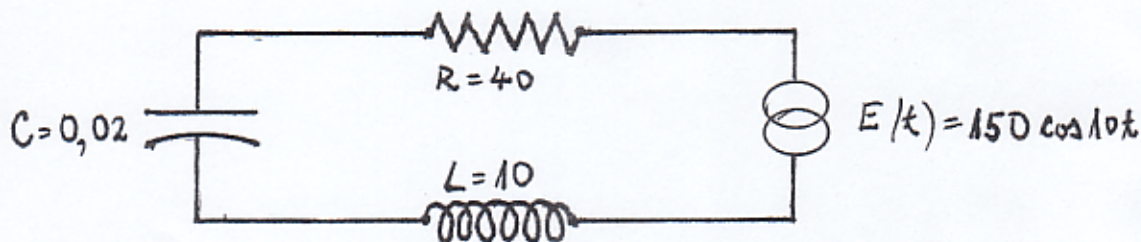
$$y'' - 4y = x^{-1} e^{2x}.$$

(10)#7... A mass of 4 kg is attached to a spring with spring constant 36 N/m



and the entire system is immersed in a liquid whose damping is 24 times the velocity. Find the equation of motion of this spring system if it is released 2 m below the equilibrium position with an upward velocity of 12 m/sec., where m stands for the metric unit of meter. system.

(10)#8... Given the  $L$ - $R$ - $C$  circuit



with charge  $Q = Q(t)$  satisfying the linear second order ordinary differential equation

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t) \quad \text{where: } R = 40 \text{ ohms, } L = 10 \text{ henries, } C = 0.02 \text{ farads}$$

and an electromotive force  $E(t) = 150 \cos 10t$  volts. Charge  $Q$  further satisfies the initial conditions  $Q(0) = 1500$  volts and  $Q'(0) = 0$ .

(A) Solve for the charge  $Q(t)$ .

(B) Indicate in  $Q(t)$  the transient and steady state terms.

(10)#9...Find the general solution of the linear system of ordinary differential equations

$$\begin{aligned} (d/dt + 1)x + (d/dt - 1)y &= 2 \\ 3x + (d/dt + 2)y &= -1 \end{aligned}$$

where  $d/dt$  is the derivative with respect to time  $t$ .

(10)#10. Find <sup>two</sup> linearly independent power series solutions  $\phi_1(x)$  and  $\phi_2(x)$  of the second order ordinary differential equation

$$y'' + 2xy' + y = 0,$$

assuming that the solution has the power series form  $y = y(x) = \sum_{n=0}^{\infty} A_n x^n$ .

Calculate only the first three non-zero terms of each of the functions  $\phi_1(x)$  and  $\phi_2(x)$ .

Solutions to the Final Exam - Winter 2007

① (A)  $\frac{dy}{dx} = \frac{y^2 e^x}{1+e^{2x}}$  is a separable ODE.

$$\frac{1}{y^2} dy = \frac{e^x dx}{1+e^{2x}}$$

$$\int \frac{1}{y^2} dy = \int \frac{e^x}{1+(e^x)^2} dx \quad \begin{array}{l} u = e^x \\ du = e^x dx \end{array}$$

$$-\frac{1}{y} = \int \frac{1}{1+u^2} du$$

$$-\frac{1}{y} = \arctan(e^x) + C \quad \begin{array}{l} \text{general solution of the ODE} \\ \text{in implicit form} \\ C = \text{constant} \end{array}$$

(B)  $\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos x$  is a linear ODE

$$\mu(x) = e^{-\int \frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2} \text{ is an integrating factor}$$

$$x^{-2} \frac{dy}{dx} - 2x^{-3}y = \cos x$$

$$\frac{d}{dx}(x^{-2}y) = \cos x \Rightarrow x^{-2}y = \int \cos x dx$$

$$\Rightarrow y = x^2(\sin x + C), \quad C = \text{constant}$$

Furthermore, we want the constant  $C$  such that  $y(\pi) = 4$ .

$$\therefore 4 = \pi^2 (\sin \pi + C)$$

$$4 = \pi^2 C \quad \Rightarrow \quad C = \frac{4}{\pi^2}$$

$$\therefore \text{Final answer: } y = x^2 \left( \sin x + \frac{4}{\pi^2} \right) \quad \#$$

② Note that  $\frac{\partial}{\partial x} (4y + 9x^2) = 18x \neq \frac{\partial}{\partial y} (6xy) = 6x$

We calculate  $\frac{M_y - N_x}{N} = \frac{\frac{\partial}{\partial y} (6xy) - \frac{\partial}{\partial x} (4y + 9x^2)}{4y + 9x^2} =$

$$= \frac{6x - 18x}{4y + 9x^2} = \frac{-12x}{4y + 9x^2} \quad \text{not a function of } x \text{ only}$$

and  $\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$  a function of  $y$

only  $\Rightarrow$  set up:  $\frac{\mu'(y)}{\mu(y)} = \frac{2}{y} \Rightarrow \ln \mu(y) = 2 \ln y + C$   
(take  $C=0$ ).

$\Rightarrow$  An integrating factor for this eq. is  $\ln \mu(y) = 2 \ln y = \ln y^2$   
i.e.  $\mu(y) = y^2$

We multiply by  $y^2$  the ODE to obtain:

$$6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0 \quad (\text{This ODE is exact.})$$

Thus, we seek a function of two variables  $f(x, y)$  such that

$$\begin{cases} \frac{\partial f}{\partial x} = 6xy^3 \\ \frac{\partial f}{\partial y} = 4y^3 + 9x^2y^2 \end{cases} \Rightarrow f(x, y) = 3x^2y^3 + c(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 9x^2y^2 + c'(y) = 4y^3 + 9x^2y^2 \Rightarrow c'(y) = 4y^3$$

$$\text{Thus } c(y) = y^4 + c, \text{ so } f(x, y) = 3x^2y^3 + y^4 + c$$

$c = \text{constant}$

Hence the solution to the given ODE, in implicit form, is:

$$3x^2y^3 + y^4 + c = 0, \quad c = \text{constant}$$

(3) (A)  $\frac{dy}{dx} = \frac{x+3y}{3x+y}$  is a homogeneous ODE solvable with

the substitution  $u = \frac{y}{x} \Leftrightarrow y = ux$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot x + u$$

$$x \frac{du}{dx} + u = \frac{x + 3ux}{3x + ux}$$

$$x \frac{du}{dx} = \frac{x(1+3u)}{x(3+u)} - u$$

$$x \frac{du}{dx} = \frac{1+3u}{3+u} - u \quad \text{or} \quad x \frac{du}{dx} = \frac{1+3u-3u-u^2}{3+u}$$

This ODE in  $u$  and  $x$  is separable:

$$\frac{3+u}{1-u^2} du = \frac{1}{x} dx$$

Use partial fractions decomposition to evaluate  $\int \frac{3+u}{1-u^2} du$ .

$$\frac{3+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$3+u = A(1+u) + B(1-u)$$

$$u=1: 4 = 2A \Rightarrow A=2$$

$$u=-1: 2 = 2B \Rightarrow B=1$$

$$\therefore \frac{3+u}{1-u^2} = \frac{2}{1-u} + \frac{1}{1+u}$$

$$\text{and } \int \frac{3+u}{1-u^2} du = 2 \int \frac{1}{1-u} du + \int \frac{1}{1+u} du =$$

$$= 2 \cdot (-1) \ln |1-u| + \ln |1+u| + C$$

$$= -2 \ln |1-u| + \ln |1+u| + C$$

$$\therefore -2 \ln \left| 1 - \frac{y}{x} \right| + \ln \left| 1 + \frac{y}{x} \right| + C = \ln x, \quad C = \text{constant}$$

is the general solution of the given ODE in implicit form  
(this form can be simplified further:

$$\ln \left( \left| 1 - \frac{y}{x} \right|^{-2} \cdot \left| 1 + \frac{y}{x} \right|^{-1} \cdot x \right) = C \Leftrightarrow \left| 1 - \frac{y}{x} \right|^2 \cdot x = C \left| 1 + \frac{y}{x} \right|$$

$C = \text{constant, etc.}$ )

(B)  $\frac{dy}{dx} + y = 8xy^4$  is a Bernoulli equation with  $n=4$

$\Rightarrow$  We'll use the substitution  $u = y^{-3}$

$$\frac{du}{dx} = -3y^{-4} \frac{dy}{dx}$$

$$-3y^{-4} \frac{dy}{dx} - 3y^{-3} = -24x$$

$\frac{du}{dx} - 3u = -24x$  is linear in  $u(x)$ .

$$\Rightarrow \mu(x) = e^{-3x} \Rightarrow e^{-3x} u' - 3e^{-3x} u = -24x e^{-3x}$$

$$\Rightarrow (e^{-3x} u)' = -24x e^{-3x}$$

$$\Rightarrow e^{-3x} u = -24 \int x e^{-3x} dx$$

We'll use integration by parts for the last integral:

$$\begin{aligned} u &= x & \Rightarrow du &= dx \\ dv &= e^{-3x} dx & v &= -\frac{1}{3} e^{-3x} \end{aligned}$$

$$\Rightarrow e^{-3x} u = -24 \left( -\frac{1}{3} x e^{-3x} + \int \frac{1}{3} e^{-3x} dx \right)$$

$$e^{-3x} u = 8x e^{-3x} + \frac{8}{3} e^{-3x} + c$$

$$\Rightarrow y^{-3} = 8x + \frac{8}{3} + c e^{3x} \Rightarrow y = \left( 8x + \frac{8}{3} + c e^{3x} \right)^{-3}$$

$c = \text{constant}$ .

#

(4) Let  $q(t)$  denote the quantity of salt at time  $t$

$$\text{So: } q(0) = 500 \text{ (kg)}$$

$$\frac{dq}{dt} = \text{Rate (in)} - \text{Rate (out)}$$

$$\frac{dq}{dt} = 10 \cdot 0,3 - 10 \frac{q}{10,000}$$

$$\frac{dq}{dt} = 3 - \frac{q}{1000}$$

$$\frac{dq}{dt} + \frac{q}{1000} = 3 \text{ linear ODE}$$

$$\mu(t) = e^{\int \frac{1}{1000} dt} = e^{\frac{t}{1000}}$$

$$(e^{\frac{t}{1000}} \cdot q)' = 3 e^{\frac{t}{1000}}$$

$$e^{\frac{t}{1000}} \cdot q = \frac{3}{\frac{1}{1000}} \cdot e^{\frac{t}{1000}} + c$$

$$q(t) = 3,000 + c e^{-t/1,000}$$

$$500 = q(0) = 3,000 + c \Rightarrow c = -2,500$$

$$\text{So (A): } q(t) = 3,000 - 2,500 \cdot e^{-t/1,000}$$

(B) as  $t \rightarrow \infty$ ,  $e^{-t/1,000} \rightarrow 0$ , so  $q(t) \rightarrow 3,000 \text{ kg}$

#

⑤ let  $y = x^m$ ,  $y' = mx^{m-1}$ ,  $y'' = m(m-1)x^{m-2}$

$x^2 y'' - 6xy' - 18y = 0$  is a Cauchy-Euler eq.

with the auxiliary equation:

$$m(m-1) - 6m - 18 = 0$$

$$m^2 - 7m - 18 = 0 \quad \text{its roots are} \quad \frac{7 \pm \sqrt{49 + 72}}{2} = \frac{7 \pm 11}{2} \begin{matrix} 9 \\ -2 \end{matrix}$$

Hence  $y_{\text{gen}}(x) = C_1 x^{-2} + C_2 x^9$

$$y(1) = 2 \Rightarrow C_1 + C_2 = 2 \quad | \cdot 2$$

$$y'(1) = 16 \Rightarrow -2C_1 + 9C_2 = 16$$

$$\Rightarrow \begin{cases} 2C_1 + 2C_2 = 4 \\ -2C_1 + 9C_2 = 16 \end{cases}$$

$$\hline \quad \quad \quad 11C_2 = 20 \Rightarrow C_2 = \frac{20}{11}$$

$$C_1 = 2 - C_2 = 2 - \frac{20}{11} = \frac{2}{11}$$

Answer:  $y(x) = \frac{2}{11} x^{-2} + \frac{20}{11} x^9$  #

⑥ Consider first:  $y'' - 4y = 0$  with  $r^2 - 4 = 0 \Rightarrow r = \pm 2$

$$\Rightarrow y_c(x) = C_1 e^{2x} + C_2 e^{-2x}$$

$$W(e^{2x}, e^{-2x}) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4.$$

$$y_p(x) = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x) = u_1 \cdot e^{2x} + u_2 e^{-2x}$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-2x} \\ x^{-1}e^{2x} & -2e^{-2x} \end{vmatrix}}{W} = \frac{1}{4} \cdot x^{-1}$$

$$u_1 = \frac{1}{4} \int \frac{1}{x} dx = \frac{1}{4} \ln x, \quad \text{we take } c=0.$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & x^{-1}e^{2x} \end{vmatrix}}{W} = -\frac{1}{4} \cdot x^{-1} e^{4x}$$

$$u_2 = -\frac{1}{4} \int \frac{1}{x} e^{4x} dx \quad (\text{and this is the best one can do!})$$

$$\Rightarrow y_p(x) = \left(\frac{1}{4} \ln x\right) \cdot e^{2x} + \left(-\frac{1}{4} \int \frac{1}{x} e^{4x} dx\right) e^{-2x}$$

Answer:  $y_{gen}(x) = y_c(x) + y_p(x)$ ,  $C_1, C_2 = \text{constants}$  #

(7)

$$m x'' = -kx - 24x'$$

$$4x'' + 24x' + 36x = 0$$

$$x(0) = 2 \quad (\text{m})$$

$$x'(0) = -12 \quad (\text{m/sec})$$

$$4r^2 + 24r + 36 = 0 \Leftrightarrow r^2 + 6r + 9 = 0 \text{ has the double root } r = -3$$

$$\text{So } x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

$$x(0) = 2: c_1 + 0 = 2 \Rightarrow c_1 = 2$$

$$x'(t) = -3c_1 e^{-3t} - 3c_2 t e^{-3t} + c_2 e^{-3t}$$

$$x'(0) = -3c_1 + c_2 = -12$$

$$-6 + c_2 = -12, \quad c_2 = -6$$

$$\therefore x(t) = 2e^{-3t} - 6te^{-3t} \quad \#$$

(8)

$$10Q'' + 40Q' + \frac{100}{2}Q = 150 \cos(10t)$$

$$10Q'' + 40Q' + 50Q = 0 \Rightarrow r^2 + 4r + 5 = 0$$

$$\text{with roots } \frac{-4 \pm \sqrt{16-25}}{2} = \frac{-4 \pm 3i}{2}$$

$$\therefore Q(t) = c_1 e^{-2t} \cos\left(\frac{3t}{2}\right) + c_2 e^{-2t} \sin\left(\frac{3t}{2}\right)$$

We seek  $Q_p(t)$  of the form  $Q_p(t) = A \cos(10t) + B \sin(10t)$

$$Q_p'(t) = -10A \sin(10t) + 10B \cos(10t)$$

$$Q_p''(t) = -100A \cos(10t) - 100B \sin(10t)$$

$$\Rightarrow -100A \cos(10t) - 100B \sin(10t) + 40A \sin(10t) + 40B \cos(10t)$$

$$+ 5A \cos(10t) + 5B \sin(10t) = 15 \cos(10t)$$

$$(-100A + 40B + 5A) \cos(10t) + (-100B - 40A + 5B) \sin(10t) = 15 \cos(10t)$$

$$\therefore \begin{cases} -95A + 40B = 15 \\ -40A - 95B = 0 \end{cases} \Rightarrow A = -\frac{95B}{40}$$

$$\frac{95^2}{40} B + 40B = 15 \Rightarrow B \left( \frac{95^2 + 40^2}{40} \right) = 15 \Rightarrow B = \frac{15 \cdot 40}{95^2 + 40^2} = \frac{600}{10,625}$$

$$A = -\frac{95}{40} \cdot \frac{15 \cdot 40}{95^2 + 40^2} = -\frac{95 \cdot 15}{95^2 + 40^2} = -\frac{1,425}{10,625}$$

$$\therefore (A) \underset{\text{gen}}{q(t)} = c_1 e^{-2t} \cos \frac{3t}{2} + c_2 e^{-2t} \sin \frac{3t}{2} - \frac{1,425}{10,625} \cos(10t) + \frac{600}{10,625} \sin(10t)$$

$$Q(0) = c_1 - \frac{1,425}{10,625} = 1,500 \Rightarrow c_1 = 1,500 + \frac{1,425}{10,625}$$

$$Q'(t) = c_1 \left( -2e^{-2t} \cos \frac{3t}{2} + e^{-2t} \cdot \left( -\sin \frac{3t}{2} \right) \cdot \frac{3}{2} \right)$$

$$+ c_2 \left( -2e^{-2t} \sin \frac{3t}{2} + e^{-2t} \cdot \cos \frac{3t}{2} \cdot \frac{3}{2} \right)$$

$$+ \frac{14,250}{10,625} \sin(10t) + \frac{6,000}{10,625} \cos(10t)$$

$$Q'(0) = -2c_1 + \frac{3}{2}c_2 + \frac{6,000}{10,625} = 0$$

$$c_2 = \frac{2}{3} \left( 2c_1 - \frac{6,000}{10,625} \right) = \frac{2}{3} \left( 3,000 + \frac{2,850 - 6,000}{10,625} \right)$$

$$= \frac{2}{3} \left( 3,000 - \frac{3,150}{10,625} \right)$$

$$= 2,000 - \frac{2,100}{10,625}$$

Finally

$$Q(t) = \left( 1,500 + \frac{1,425}{10,625} \right) e^{-2t} \cos \frac{3t}{2} + \left( 2,000 - \frac{2,100}{10,625} \right) e^{-2t} \sin \frac{3t}{2} - \frac{1,425}{10,625} \cos(10t)$$

$$+ \frac{600}{10,625} \sin(10t)$$

(B) The transient terms are the 1st two (containing  $e^{-2t}$  which approaches zero as  $t \rightarrow \infty$ ).  
The last two terms are steady-state terms. #

$$(9) \begin{cases} (D+1)x + (D-1)y = 2 \\ 3x + (D+2)y = -1 \end{cases} \quad \begin{array}{l} -3 \\ (D+1) \end{array}$$

$$\begin{cases} -3(D+1)x - 3(D-1)y = -6 \\ 3(D+1)x + (D+1)(D+2)y = -1 \end{cases}$$

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$$✓ (D^2 + 3D + 2 - 3D + 3)y = -7$$

$$(D^2 + 5)y = -7$$

$$y'' + 5y = -7$$

$$y'' + 5y = 0 \Rightarrow r^2 + 5 = 0 \Rightarrow r = \pm i\sqrt{5}$$

$$y_c(t) = C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t)$$

$$\begin{cases} y_p(t) = A \\ y_p'(t) = 0 \\ y_p''(t) = 0 \end{cases} \Rightarrow \begin{cases} 5A = -7 \\ A = -7/5 \end{cases}$$

$$\therefore \underset{\text{gen}}{y}(t) = C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t) - \frac{7}{5}$$

$$\underset{\text{gen}}{x} = \frac{1}{3} (-1 - (D+2)y) = -\frac{1}{3} - \frac{1}{3} (y' + 2y) =$$

$$= -\frac{1}{3} - \frac{1}{3} (-\sqrt{5}C_1 \sin(\sqrt{5}t) + C_2 \sqrt{5} \cos(\sqrt{5}t) + 2C_1 \cos(\sqrt{5}t) + 2C_2 \sin(\sqrt{5}t) - \frac{14}{5})$$

Answer:

$$\begin{cases} x(t) = \left( -\frac{\sqrt{5}}{3} c_2 - \frac{2c_1}{3} \right) \cos(\sqrt{5}t) + \left( \frac{\sqrt{5}}{3} c_1 - \frac{2}{3} c_2 \right) \sin(\sqrt{5}t) + \frac{3}{5} \\ y(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) - \frac{7}{5} \end{cases} \neq$$

⑩ Let  $y(x) = \sum_{n=0}^{\infty} A_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} n A_n x^{n-1} = \sum_{n=1}^{\infty} n A_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) A_n x^{n-2}$$

$$\therefore \sum_{n=2}^{\infty} n(n-1) A_n x^{n-2} + 2 \sum_{n=1}^{\infty} n A_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} x^n + 2 \sum_{n=0}^{\infty} n A_n x^n + \sum_{n=0}^{\infty} A_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) A_{n+2} + 2n A_n + A_n \right] x^n = 0$$

$$\therefore A_{n+2} = -\frac{(2n+1)}{(n+2)(n+1)} A_n$$

$$\begin{array}{l|l} n=0: A_2 = -\frac{1}{2} A_0 & n=1: A_3 = -\frac{3}{3 \cdot 2} A_1 = -\frac{1}{2} A_1 \\ n=2: A_4 = -\frac{5}{4 \cdot 3} A_2 = \frac{5}{4 \cdot 3 \cdot 2} A_0 & n=3: A_5 = -\frac{7}{5 \cdot 4} A_3 = \frac{7}{5 \cdot 4 \cdot 2} A_1 \end{array}$$

$$\Rightarrow y(x) = A_0 \left( 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \dots \right) + A_1 \left( x - \frac{x^3}{2} + \frac{7}{40}x^5 - \dots \right)$$

$$\text{So: } \varphi_1(x) = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 - \dots$$

$$\text{and } \varphi_2(x) = x - \frac{x^3}{2} + \frac{7}{40}x^5 - \dots$$

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