Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1341A – The Midterm Test II (v.1) **Instructor: K. Zaynullin**

Last name: ______________________

First name: ______________________

Student number:

Please, read the following instructions carefully:

- You have 80 minutes to complete this test. **Do not detach** the pages of this examination. Read each question carefully. Where it is possible to check your work, do so.
- You can use the backs of the pages and the last page for computations.
- This is a closed book exam, and no notes of any kind are allowed. The use of programmable calculators, cell phones, laptops, pagers or any text storage or communication device is not permitted.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	Total
Mark								
Out of	1	1	1	1	3	5	8	20

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1. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid xy + zw = 0\}$. Then, (1)

`cross (X) the correct answer:`

- [A] U is closed under addition and U is closed under multiplication by scalars
- [B] $(0, 0, 0, 0) \in U$ and U is closed under addition
- [C] U is not closed under addition and U is not closed under multiplication by scalars
- [D] U is closed under addition but U is not closed under multiplication by scalars
- [E] $(0, 0, 0, 0) \in U$ and U is closed under multiplication by scalars
- [F] None of the other statements is true

Solution: The correct answer is [E].

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2. Which of the following are subspaces of $\mathbb{R}^3$? (1)

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y + z = 0\}$$

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid (x - z)(x - y) = 0\}$$

$$W = \{(x + y, 2y, x - y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

$$X = \{(x, xy, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

`cross (X) the correct answer:`

- [A] Only U and V
- [B] Only U and W
- [C] Only U and X
- [D] Only V and W
- [E] Only V and X
- [F] Only W and X

Solution: U is a plane through 0, so it is a linear subspace.

V is a union of two different planes, so it is NOT a linear subspace.

W is a linear span of $(1, 0, 1)$ and $(1, 2, -1)$, so it is a linear subspace

X is not a plane, so it is NOT a linear subspace.

Hence, the correct answer is [B].

3. It is known that a subspace Y of $\mathbb{R}^{99}$ can be spanned by 75 vectors and that Y has a linearly independent set with 63 vectors. Then it is always true that: (1)

`cross (X) the correct answer:`

- [A] $63 < \dim Y < 99$
- [B] $63 \leq \dim Y < 75$
- [C] $63 \leq \dim Y \leq 99$
- [D] $75 \leq \dim Y \leq 99$
- [E] $\dim Y < 63$
- [F] $\dim Y > 63$

Solution: We know that the size of any linearly independent subset in Y is less or equal than the dimension of Y and the dimension of Y does not exceed the size of any spanning set of Y. So the correct answer is [C].

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4. Suppose $\{u, v\}$ is a linearly independent set in vector space V, and that $w \in V$ is chosen so that $\{u, v, w\}$ is linearly independent. Which of the following statements is ALWAYS true?

(1)

`cross (X) the correct answer:`

- [A] $\{u, w\}$ is linearly independent
- [B] $\{v, w\}$ is linearly dependent
- [C] $u \in Span\{v, w\}$
- [D] $v \in Span\{u\}$
- [E] $w \notin Span\{v, w\}$
- [F] $V = Span\{u, v, w\}$

Solution: The correct answer is [A].

5. Consider the vector space $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomial functions of degree at most 2, and define

$$W = \{p(x) \in P_2 \mid p(0) = 0\}.$$

5a) Show that W is a linear subspace of P_2. (1)

Solution: Use the subspace test:

$0(x) \in W$.

If $p, q \in W$, then $(p + q)(0) = (p + q)(0) = p(0) + q(0) = 0 + 0 = 0$, so $p + q \in W$.

Finally, if $p \in W$ and $c \in \mathbb{R}$, then $(cp)(0) = cp(0) = c0 = 0$, so $cp \in W$.

5b) Find a basis for W. (2)

a correct answer without explanations is (1) only

Solution: We have

$$W = \{a + bx + cx^2 \in P_2 \mid a, b, c \in \mathbb{R},\ a = 0\} = \{bx + cx^2 \mid b, c \in \mathbb{R}\}$$

Since $\{x, x^2\}$ is a linearly independent subset and it spans W, it is a basis of W.

6. Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{pmatrix} a & b \\ a-b & a+c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b, c \in \mathbb{R} \right\}$$

6a) Show that U is a linear subspace of $M_{2\times 2}(\mathbb{R})$. (1)

Solution: We show that

$$U = Span \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Any span is always a linear subspace.

6b) Find a basis for U, and, hence, find $\dim U$. (2)

Solution: We show that the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

are linearly independent, hence, they form a basis of U and $\dim U = 3$.

6c) Give a basis for U different from the one you gave in (b). (2)

This has to be a linearly independent spanning set for the same U. Replace the first matrix by the sum of the first and the second. We get a different basis

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

7. State whether each of the following statements is always true, or is possibly false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example where it fails.
- If you say the statement is always true, you must give a clear explanation.

7a) $X = \{f \in F(\mathbb{R}) \mid f(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$ is a subspace of $F(\mathbb{R}) = \{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$.

Answer: ☐ (1)

Justification: (1)

Solution: NO, as it is not closed under multiplication by a scalar:

Take $f(x) = x^2$. Then $f \in X$, but $(-1)f(x) = -x^2 \leq 0$ so $(-1) \cdot f \notin X$.

7b) $U = \left\{ \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b, c \in \mathbb{R} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$ of dimension 3.

Answer: ☐ (1)

Justification: (1)

Solution: No, as the zero-matrix is not in U

7c) If v_1, v_2, v_3 are non-zero vectors in a vector space V, and $U = Span\{v_1, v_2, v_3\}$, then $\dim U = 3$.

Answer: (1)

Justification: (1)

Solution: No, as we can take $v_1 = v_2 = v_3 = (1, 1)$ and $\dim U = 1$.

7d) Let $U = Span\{x\sin^2(x), x\cos^2(x), x\}$ be a subspace of $F(\mathbb{R})$. Then $\dim U = 3$.

Answer: (1)

Justification: (1)

Solution: No, as $x = x\sin^2(x) + x\cos^2(x)$, so the set $\{x\sin^2(x), x\cos^2(x), x\}$ is linearly dependent. Hence, $\dim U < 3$.

The last page (use it for computations)