Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1341E – Final Exam

Instructor: S. Molladavoudi

LAST NAME: KEY

FIRST NAME:

STUDENT NUMBER:

SEAT NUMBER:

Please, read the following instructions carefully:

- **You have 3 hours to complete this test.** Do not detach the pages of this examination. Read each question carefully. For rough work you may use the back pages.
- This is a closed book exam, and no notes of any kind are allowed. The use or possession at your exam desk of programmable calculators, cell phones, laptops, pagers or any text storage or communication device is not permitted. **By signing the attendance sheet you acknowledge that you will comply with these conditions.**
- Questions 1 to 10 are multiple choice. These questions are worth **1 point** each and no part marks will be given.
- Questions 11 – ~~14~~ require a complete solution. Answer these questions in the space provided. **Remember that the correct answer requires justification written legibly and logically.**

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
Mark	C	D	F	F	C	F	D	C	D	B						
Out of	1	1	1	1	1	1	1	1	1	1	2	3	4	4	2	25

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

1. Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid x+y+z = 0\}$. Which one of the following statements is true? (1)

mark (X) the correct answer:

- $\boxed{\text{A}}$ U is not a subspace of $\mathbb{R}^3$
- $\boxed{\text{B}}$ U is a subspace of $\mathbb{R}^3$ and $\dim(U) = 3$
- $\boxed{\cancel{\text{C}}}$ U is a subspace of $\mathbb{R}^3$ and $\{(1, 0, -1), (0, 1, -1)\}$ is a basis of U
- $\boxed{\text{D}}$ U is a line in $\mathbb{R}^3$ with direction vector $(1, 1, 1)$
- $\boxed{\text{E}}$ U is a subspace of $\mathbb{R}^3$ and $\{(1, 0, -1), (-1, 0, 1)\}$ is a basis of U
- $\boxed{\text{F}}$ U is a plane in $\mathbb{R}^3$ with normal vector $(-1, 2, -1)$

$U = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\} \Rightarrow z = -x-y$

$U = \{(x, y, -x-y) \mid x, y \in \mathbb{R}\}$

$U = \{x(1,0,-1) + y(0,1,-1) \mid x, y \in \mathbb{R}\}$

$U = \text{Span}\{(1,0,-1), (0,1,-1)\}$ is a subspace

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2. Which two of the following statements are true? (1)

✗ I. $\{1, \sin^2 x, \cos^2 x\}$ is a linearly independent set of vectors in the vector space of all real-valued functions $F[\mathbb{R}] = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. $1 = \sin^2 x + \cos^2 x \Rightarrow 1 \in \text{Span}\{\sin^2 x, \cos^2 x\}$

✓ II. A homogeneous system of linear equations is always consistent. trivial solution

✓ III. If A and B are 3×3 matrices, and both A and B are invertible, then their product AB is also an invertible matrix. If A & B are invertible $\Rightarrow AB$ is invertible too

✗ IV. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent vectors in $\mathbb{R}^3$, then $\dim(\text{Span}\{\mathbf{u}, \mathbf{v}\}) = 2$.

$\vec{v} = a\vec{u}$

$\Rightarrow \dim(\text{span}\{\vec{u}, a\vec{u}\}) = 1$

mark (X) the correct answer:

- $\boxed{\text{A}}$ I. and II.
- $\boxed{\text{B}}$ I. and III.
- $\boxed{\text{C}}$ I. and IV.
- $\boxed{\cancel{\text{D}}}$ II. and III.
- $\boxed{\text{E}}$ II. and IV.
- $\boxed{\text{F}}$ III. and IV.

3. Let A be a square $n \times n$ matrix ($n \geq 2$). Which of the following statements are true? (1)

✗ I. If rank(A) = 1, there is just one parameter in the general solution of the system $A\mathbf{x} = \mathbf{0}$.

✓ II. If rank(A) = 1, there are $n-1$ parameters in the general solution of the system $A\mathbf{x} = \mathbf{0}$.

✗ III. If A is invertible, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

✓ IV. If the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then rank(A) $< n$.

mark (X) the correct answer:

- $\boxed{\text{A}}$ I. only
- $\boxed{\text{B}}$ II. only
- $\boxed{\text{C}}$ I. and III.
- $\boxed{\text{D}}$ II. and III.
- $\boxed{\text{E}}$ I. and IV.
- $\boxed{\text{F}}$ (X) II. and IV.

Rank-nullity theorem:

rank(A) + # parameters = # columns = n

if rank(A) = 1 ⇒ # parameters = n − 1

if A is invertible, then $A\vec{x} = \vec{0}$ has a unique solution ($\vec{x} = \vec{0}$)

if $A\vec{x} = \vec{0}$ has parameters in its general solution, then rank(A) < n.

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4. Find the polar form of the complex number $\frac{1-\sqrt{3}i}{i-1}$. (1)

(You can find the table of trigonometric functions in the last page.)

mark (X) the correct answer:

- $\boxed{\text{A}}$ $\sqrt{2}(\cos(-7\pi/12) + i\sin(-7\pi/12))$
- $\boxed{\text{B}}$ $\sqrt{2}(\cos(5\pi/12) + i\sin(5\pi/12))$
- $\boxed{\text{C}}$ $\sqrt{2}(\cos(-\pi/12) + i\sin(-\pi/12))$
- $\boxed{\text{D}}$ $\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$
- $\boxed{\text{E}}$ $\sqrt{2}(\cos(-5\pi/12) + i\sin(-5\pi/12))$
- $\boxed{\text{F}}$ (X) $\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$

$z_1 = a_1 + b_1 i = 1 - \sqrt{3}i$

$r_1 = |z_1| = \sqrt{1+3} = 2$

$\left.\begin{aligned} \sin(\theta_1) &= \frac{b_1}{r_1} = -\frac{\sqrt{3}}{2} \\ \cos(\theta_1) &= \frac{a_1}{r_1} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \theta_1 = -\pi/3$

However, we know that $\theta_1' = \theta_1 + 2\pi$ would work too $\Rightarrow \theta_1' = 5\pi/3$

$|z_2| = \sqrt{1+1} = \sqrt{2}$, $\sin(\theta_2) = \frac{1}{\sqrt{2}}$, $\cos(\theta_2) = -\frac{1}{\sqrt{2}} \Rightarrow \theta_2 = 3\pi/4$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1' - \theta_2)} = \frac{2}{\sqrt{2}} e^{i(5\pi/3 - 3\pi/4)} = \sqrt{2}\, e^{i(11\pi/12)}$

5. Let $B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and consider the subset $U = \{A \in M_{2\times 2}(\mathbb{R}) \mid BA = -AB\}$.

Which one of the following statements is true? (1)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

mark (X) the correct answer:

- [A] U is not a subspace of the vector space of 2×2 real matrices $M_{2\times 2}(\mathbb{R})$
- [B] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim(U) = 0$
- [X] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim(U) = 1$
- [D] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim(U) = 2$
- [E] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim(U) = 3$
- [F] U is a subspace of $M_{2\times 2}(\mathbb{R})$, and $\dim(U) = 4$

$$-AB = -\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a = 0 \\ c = 0 \\ b = d \end{cases}$$

$$\Rightarrow U = \left\{ \begin{bmatrix} 0 & b \\ 0 & b \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \mid b \in \mathbb{R} \right\} : \text{subspace \& } \dim(U) = 1.$$

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6. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, then the third row of A^{-1} is: (1)

mark (X) the correct answer:

- [A] [0 1 0]
- [B] [1 0 2]
- [C] [1 0 − 1]
- [D] [0 0 1]
- [E] [1 0 1]
- [X] [−1 1 1]

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{-R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & \boxed{-1 \quad 1 \quad 1} \end{array}\right]$$

other operations to find A^{-1} will not change the 3rd row.

7. For a non-homogeneous system of 13 equations in 15 unknowns, answer the following three questions (Yes/No): (1)

$$13\left[\begin{array}{c|c} \overset{15}{0} & \begin{matrix}1\\0\\ \vdots \\0\end{matrix}\end{array}\right] \text{ inconsistent}$$

(a) • Can the system be inconsistent? Yes !

(b) • Can the system have a unique solution? NO,

(c) • Can the system have infinitely many solutions? Yes

mark (X) the correct answer:

- [A] Yes, Yes, Yes
- [B] Yes, Yes, No
- [C] No, Yes, Yes
- [X] Yes, No, Yes
- [E] No, No, Yes
- [F] Yes, No, No

(b) The rank is at most 13, so there are at least 2 parameters in the general solution of $A\vec{x}=\vec{b}$ (rank-nullity theorem)

(c) Again, according to rank-nullity theorem the system $A\vec{x}=\vec{b}$ will have at least 2 parameters in the general solution.

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8. Let A be a square $n \times n$ matrix.

Which one of the statements below is **not equivalent** to the statement (1)

"The columns of A are linearly independent".

mark (X) the correct answer:

- [A] The rows of A are linearly independent. ✓
- [B] $\text{rank}(A) = n$ ✓
- [X] $\det(A) \neq 1$ ✗ if $\text{rank}(A)=n \Leftrightarrow A$ invertible $\Leftrightarrow \det(A)\neq 0$ ($\det(A)$ could be 1)
- [D] The rows of A form a basis of $\mathbb{R}^n$. ✓
- [E] The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution. ✓ trivial solution ($\vec{x}=\vec{0}$)
- [F] A is invertible. ✓

9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$ for some numbers $a, b, c, d, e, f, g, h, j$ be such that $\det(A) = 2$.

Using the elementary row/column operations compute the determinant of the matrix (1)

$$\begin{bmatrix} b+5c & e+5f & h+5j \\ 3a & 3d & 3g \\ -2c & -2f & -2j \end{bmatrix}$$

mark (X) the correct answer:

- [A] -64
- [B] -18
- [C] -12
- [X] 12
- [E] 18
- [F] 64

$$\begin{vmatrix} b+5c & e+5f & h+5j \\ 3a & 3d & 3g \\ -2c & -2f & -2j \end{vmatrix} = 3(-2) \begin{vmatrix} b+5c & e+5f & h+5j \\ a & d & g \\ c & f & j \end{vmatrix}$$

$$\underset{R_1 \leftrightarrow R_2}{=} (-6)(-1) \begin{vmatrix} a & d & g \\ b+5c & e+5f & h+5j \\ c & f & j \end{vmatrix} = 6 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & j \end{vmatrix}$$

$$\underset{\text{transpose}}{=} 6 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 6(2) = 12$$

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10. The vectors $\mathbf{u}_1 = (1, -1, 2)$, $\mathbf{u}_2 = (-5, -1, 2)$, and $\mathbf{u}_3 = (0, 2, 1)$ form an orthogonal basis of $\mathbb{R}^3$. We know that any vector $\mathbf{v} \in \mathbb{R}^3$ can be expressed in a unique way as a linear combination $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$, where $a_1, a_2, a_3 \in \mathbb{R}$ are the coordinates (or the Fourier coefficients) of $\mathbf{v}$ with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Find the coordinate a_2 for $\mathbf{v} = (1, 0, 1)$. (1)

mark (X) the correct answer:

- [A] -0.2
- [X] -0.1
- [C] -1
- [D] 1
- [E] 0.1
- [F] 0.2

$$a_2 = \frac{\vec{v} \cdot \vec{u}_2}{\|\vec{u}_2\|^2} = \frac{(1,0,1)\cdot(-5,-1,2)}{25+1+4}$$

$$= \frac{-5+2}{30} = \frac{-3}{30} = -\frac{1}{10} = -0.1$$

11. Let $U = \mathrm{Span}\{(-1,1,1,0),(1,0,1,1),(2,1,4,3),(0,1,2,2)\}$ in $\mathbb{R}^4$.

(a) Find a basis for U which is a subset of the given spanning set above. (1)

ANSWER: basis = $\{(-1,1,1,0),(1,0,1,1),(0,1,2,2)\}$

we use the column space algorithm:

$$A = \begin{array}{c} \begin{bmatrix} -1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \\ \begin{matrix} \vec{C_1} & \vec{C_2} & \vec{C_3} & \vec{C_4} \end{matrix} \end{array} \underset{R_2 \leftrightarrow R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\underset{\substack{R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \underset{\substack{-R_2+R_3 \to R_3 \\ -R_2+R_4 \to R_4 \\ (R_4 \leftrightarrow R_3)}}{\sim} \begin{bmatrix} 1 & 0 & \boxed{1} & 1 \\ 0 & 1 & \boxed{3} & 1 \\ 0 & 0 & \boxed{0} & 1 \\ 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

no leading 1

So, $\{\vec{C_1}, \vec{C_2}, \vec{C_4}\}$ is a basis for U.

(b) Extend the basis found in part (a) to a basis of $\mathbb{R}^4$. (1)

ANSWER: basis of $\mathbb{R}^4$ = $\{(-1,1,1,0),(1,0,1,1),(0,1,2,2),(0,0,1,0)\}$

From part (a) we conclude that there is no leading 1 in the 3rd column and if we add $(0,0,1,0)$ from the standard basis of $\mathbb{R}^4$ then we will have a leading 1 in every column and hence the columns would form a basis of $\mathbb{R}^4$.

12. Let $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid y - z - w = 0\}$.

(a) Find a basis for W. (1)

ANSWER: $\{(1,0,0,0),(0,1,0,1),(0,1,1,0)\}$

$W = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\} = \ker([0\ 1\ -1\ -1])$

$[A \mid \vec{0}] = [\underbrace{0}_{r}\ \ 1\ \ \underbrace{-1}_{s}\ \ \underbrace{-1}_{t} \mid 0] \Rightarrow x = r,\ y = s+t,\ z = s,\ w = t$

$\Rightarrow \ker(A) = W = \{(r, s+t, s, t) \mid r, s, t \in \mathbb{R}\}$

$\Rightarrow W = \{r(1,0,0,0) + s(0,1,1,0) + t(0,1,0,1) \mid r,s,t \in \mathbb{R}\}$

$\Rightarrow W = \operatorname{span}\{(1,0,0,0),(0,1,1,0),(0,1,0,1)\}$

we know that the basic solutions of kernel form a basis for W.

(b) Use the Gram-Schmidt algorithm to find an orthogonal basis for W. (1)

ANSWER: $\{(1,0,0,0),(0,1,0,1),(0,1,2,-1)\}$

let $\vec{u}_1 = (1,0,0,0)$, $\vec{u}_2 = (0,1,0,1)$, $\vec{u}_3 = (0,1,1,0)$ be the non-orthogonal basis from part (a). Then,

$\vec{\omega}_1 = \vec{u}_1 = (1,0,0,0)$

$\vec{\omega}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{\omega}_1}{\|\vec{\omega}_1\|^2}\vec{\omega}_1 = (0,1,0,1) - \frac{\overset{0}{\cancel{(0,1,0,1)\cdot(1,0,0,0)}}}{2}(1,0,0,0)$

$= (0,1,0,1)$

$\vec{\omega}_3 = \vec{u}_3 - \cancel{\frac{\vec{u}_3 \cdot \vec{\omega}_1}{\|\vec{\omega}_1\|^2}}^{0}\vec{\omega}_1 - \frac{\vec{u}_3 \cdot \vec{\omega}_2}{\|\vec{\omega}_2\|^2}\vec{\omega}_2 = (0,1,1,0) - \frac{\overset{1}{(0,1,1,0)\cdot(0,1,0,1)}}{2}\vec{\omega}$

$= (0,1,1,0) - \frac{1}{2}(0,1,0,1) = (0, \frac{1}{2}, 1, -\frac{1}{2})$ or $(0,1,2,-1)$

(c) Find the best approximation of $\mathbf{v} = (0, 1, 1, 1)$ by a vector in W. (1)

ANSWER: $\boxed{\left(0, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)}$

$$\operatorname{proj}_W(\vec{v}) = \frac{\vec{v}\cdot\vec{\omega}_1}{\|\vec{\omega}_1\|^2}\vec{\omega}_1 + \frac{\vec{v}\cdot\vec{\omega}_2}{\|\vec{\omega}_2\|^2}\vec{\omega}_2 + \frac{\vec{v}\cdot\vec{\omega}_3}{\|\vec{\omega}_3\|^2}\vec{\omega}_3$$

$$= \frac{\cancel{(0,1,1,1)\cdot(1,0,0,0)}^{\,0}}{1}(1,0,0,0) + \frac{\overbrace{(0,1,1,1)\cdot(0,1,0,1)}^{2}}{2}(0,1,0,1)$$

$$+ \frac{\overbrace{(0,1,1,1)\cdot(0,1,2,-1)}^{2}}{1+4+1}(0,1,2,-1)$$

$$\Rightarrow \operatorname{proj}_W(\vec{v}) = (0,1,0,1) + \frac{1}{3}(0,1,2,-1)$$

$$= \underline{\left(0, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)}$$

13. Let $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A. (1)

ANSWER: $\boxed{-(\lambda+2)^2(\lambda-1)}$

$$C_A(\lambda) = \det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$\underset{-R_2+R_3\to R_3}{=} \begin{vmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 0 & \lambda+2 & -2-\lambda \end{vmatrix} \underset{C_2+C_3\to C_2}{=} \begin{vmatrix} -1-\lambda & 2 & 1 \\ 1 & -\lambda & 1 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

we can now do the cofactor expansion along the third row:

$$C_A(\lambda) = (-1)^{3+3}(-2-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix}$$

$$= -(2+\lambda)(\lambda^2+\lambda-2) = -(\lambda+2)^2(\lambda-1)$$

(b) Using the characteristic polynomial explain why the eigenvalues of A are -2 and 1. (1/2)

We know that the roots of the characteristic polynomial $C_A(\lambda)$ are the eigenvalues of A, therefore

$$C_A(\lambda) = \det(A-\lambda I) = -(\lambda+2)^2(\lambda-1) = 0$$

$\Rightarrow$ eigenvalues $\begin{cases} \lambda = -2 & \text{(with algebraic multiplicity 2)} \\ \lambda = 1 & \text{(with algebraic multiplicity 1)} \end{cases}$

(c) Find a basis of the eigenspace $E_{-2} = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = -2\mathbf{v}\}$. (1)

ANSWER: $\boxed{\{(-1,1,0),(-1,0,1)\}}$

$E_{-2} = \text{Ker}(A+2I)$, where $A+2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow [A+2I \mid \vec{0}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \quad t$

$x_1 = -s-t,\ x_2 = s,\ x_3 = t$

$\text{Ker}(A+2I) = \{(-s-t, s, t) \mid t, s \in \mathbb{R}\}$
$= \text{span}\{(-1,1,0),(-1,0,1)\} \Rightarrow \dim(E_{-2}) = 2$

basic solutions are the basis of $\text{Ker}(A+2I) = E_{-2}$.

(d) Find a basis of the eigenspace $E_1 = \{\mathbf{v} \in \mathbb{R}^3 \mid A\mathbf{v} = \mathbf{v}\}$. (1)

ANSWER: $\boxed{\{(1,1,1)\}}$

$E_1 = \text{Ker}(A-I)$. where $A-I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$[A-I \mid \vec{0}] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$

$\underset{\substack{2R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s$

$x - 2y = -s,\ y = s,\ z = s \Rightarrow E_1 = \text{span}\{(1,1,1)\}\ \&\ \dim(E_1) = 1$

(e) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (1/2)

ANSWER: $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $\dim(E_{-2}) = 2 =$ algebraic multiplicity of $\lambda = -2$

& $\dim(E_1) = 1 =$ algebraic multiplicity of $\lambda = 1$

we conclude that A is diagonalizable and we can write the matrix P whose columns are the eigenvectors of A:

$$P = [\vec{V}_1 \; \vec{V}_2 \; \vec{V}_3] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$, which is invertible,

and the matrix D which is diagonal and the diagonal entries of D are the corresponding eigenvalues of A:

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. State whether each of the following is (always) **true**, or is (possibly) **false**, in the respective box. **For each statement, you <u>must</u> give a clear explanation or an explicit counter-example to prove your claim.**

(a) If A is a 5×4 matrix and if a row echelon form of A has a row of zeros, then $\text{rank}(A) < 4$.

ANSWER (True/False): $\boxed{\text{False}}$ (1/2)

Justification/counter-example: (1/2)

$$\text{RREF of } A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{5\times 4}, \text{ but } \text{rank}(A) = 4$$

(b) The dimension of the kernel of the 1×4 matrix $[-1\ 0\ 1\ 2]$ is 3.

ANSWER (True/False): $\boxed{\text{True}}$ (1/2)

Justification/counter-example: (1/2)

$\text{Ker}([-1 \ \ 0 \ \ 1 \ \ 2])$,

$$[A \mid \vec{0}] = \left[\begin{array}{cccc|c} -1 & 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \end{array}\right]$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad r \quad\ s \quad\ t.$$

From the rank-nullity theorem:

$$\#\text{ parameters} = \dim(\ker(A)) = \underset{\#\text{ columns}}{4} - \underset{\text{rank}(A)}{1} = 3$$

(c) Let z_1 and z_2 be two complex numbers which are not real numbers. Then, their product $z_1 \cdot z_2$ can not be a real number.

ANSWER (True/False): $\boxed{\text{False}}$ (1/2)

Justification/counter-example: (1/2)

suppose: $z_1 = i \in \mathbb{C}$

$z_2 = -i \in \mathbb{C}$

$\Rightarrow z_1 \cdot z_2 = z_1 \bar{z}_1 = -i^2 = 1 \in \mathbb{R}$

(d) The map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, x + y)$ is a linear transformation.

ANSWER (True/False): $\boxed{\text{True}}$ (1/2)

Justification/counter-example: (1/2)

we can check the properties of linear transformations:

suppose $\vec{u} = (x, y) \in \mathbb{R}^2$, $\vec{v} = (x', y') \in \mathbb{R}^2$ & $r \in \mathbb{R}$

$T(x+x', y+y') = (x+x', x+x'+y+y') = T(x,y) + T(x', y')$ ✓

$T(rx, ry) = (rx, rx+ry) = (rx, r(x+y))$

$= rT(x,y)$ ✓

So, T is a linear transformation.

15. Define a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

(Bonus Question)

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ y-z \\ z-x \end{bmatrix}$$

(a) Find the standard matrix of T. (1)

ANSWER: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

The standard matrix of T is obtained by

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$, where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) Find a basis for the image of T. (1)

ANSWER: $\{(1,0,-1),(0,1,-1)\}$

We know that $\operatorname{im}(T) = \operatorname{col}(A) = \operatorname{span}\{(1,0,-1),(-1,1,0),(0,-1,1)\}$

therefore, we can use the row space algorithm to find a basis for Col(A):

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

non-zero rows would form a basis for Col(A).

Last page (use it for rough work only). **Anything written on the last page will not be graded or taken into account while grading.**

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0