Calculus for the Life Sciences I

MAT1330 B Instructor: Elizabeth Maltais

Wednesday, November 8, 2017 Test #2

Duration: 75 minutes

SOLUTIONS

FAMILY NAME:
FIRST NAME:
STUDENT NUMBER:
†SIGNATURE:

† **By signing above, you acknowledge that you have carefully read, understand, and will comply with the following instructions.**

INSTRUCTIONS

- This is a **75 minute closed-book** exam; no notes are permitted. Except for Faculty-approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no electronic devices are permitted: no cell phones, no smartwatches nor related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag at the front of the classroom for the duration of the exam**.
- Read each question carefully.
- Questions 1 through 7 are multiple choice, worth a total of six points plus one bonus point. **Record your answers to the multiple choice questions in the boxes provided.**
- Questions 8 through 10 are long answer, with number of points as indicated. **You must show your work, your work must be legible and well-justified, and you must record your answers in the spaces provided.**
- Where it is possible to check your work, do so.
- Good luck!

For picking up your graded test: Circle the DGD you will attend to pick up your test (whether you are registered or not).

# :	DGD1	DGD2	DGD3	DGD4
Day :	Tuesday	Wednesday	Wednesday	Thursday
Start time:	10:00 AM	1:00 PM	1:00 PM	11:30 AM
Room :	TBT 070	LMX 390	STE J0106	FTX 361
TA :	Andrew	Zhe	Andrew	Zhe

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Marker's use only:

Question	Marks
1–7 (/6) +bonus	
8 (/5)	
9 (/5)	
10 (/8)	
Total (/24)	

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1. (1 point) If $f(x) = e^x \arctan(x)$, then $f'(1) =$

A. 0

B. $\frac{\pi e}{4}$

C. $\frac{\pi e}{4} + \frac{e}{2}$

D. $\frac{e}{2} - e\pi$

E. 1

F. e

$f'(x) = e^x \arctan(x) + e^x\left(\frac{1}{1+x^2}\right)$

$f'(1) = e^1 \arctan(1) + e^1\left(\frac{1}{1+1^2}\right)$

$= e\left(\frac{\pi}{4}\right) + e\left(\frac{1}{2}\right)$

Your answer: C

2. (1 point) Which of the following is equal to the derivative of $f(x) = 2^{\sin(x)}$?

A. $2^{\sin(x)} \cos(x)$

B. $\ln(2) 2^{\sin(x)} \cos(x)$

C. $2^{\cos(x)} \sin(x) \ln(2)$

D. $\frac{2^{\sin(x)}}{\ln(2)}$

E. $2^{\sin(x)}$

F. $\sin(x)\cos(x)2^{\sin(x)-1}$

$f'(x) = (\ln 2) \cdot 2^{\sin x} (\cos x)$

Your answer: B

3. (1 point) Which of the following is equal to the derivative of $g(x) = \ln(e^x x^3)$?

A. $\frac{e^x}{x^3}$

B. $e^x \ln(x^3) - 3e^x/x$

C. $3x^2e^x + x^3e^x$

D. $(3x^2 + x^3)e^x \ln(e^x x^3)$

E. $1 + 3/x$

F. $3/x$

$g(x) = \ln(e^x) + \ln(x^3)$

$= x + 3\ln x$

$g'(x) = 1 + 3\left(\frac{1}{x}\right)$

Your answer: E

4. (1 point) Suppose f is a function and, for all x in the domain of f, we know that

$$\tan\big(f(x)\big) + f(x) = x^2$$

What is $f'(x)$?

A. $f'(x) = \dfrac{2x}{\sec(x)\tan(x) + 1}$

B. $f'(x) = 2x - \sec(x)\tan(x)$

C. $f'(x) = \dfrac{2x}{\sec^2(x) + 1}$

D. $f'(x) = \dfrac{2x}{\sec\big(f(x)\big)\tan\big(f(x)\big) + 1}$

E. $f'(x) = 2x - \sec^2\big(f(x)\big)$

F. $f'(x) = \dfrac{2x}{\sec^2\big(f(x)\big) + 1}$

Your answer: F

$$\tan(f(x)) + f(x) = x^2$$

$$\Rightarrow \sec^2(f(x)) \cdot f'(x) + f'(x) = 2x$$

$$\Rightarrow f'(x)\left[\sec^2(f(x)) + 1\right] = 2x$$

$$\Rightarrow f'(x) = \frac{2x}{\sec^2(f(x)) + 1}$$

5. (1 point) Which of the following is the equation of the tangent line to the graph of

$$f(x) = x^{1/3} - \frac{16}{x} \qquad \text{at the point } (8, 0)\ ?$$

A. $y = \dfrac{16}{3}x - \dfrac{128}{3}$

B. $y = \dfrac{1}{3}x - \dfrac{8}{3}$

C. $y = \dfrac{2}{3}x - \dfrac{16}{3}$

D. $y = \dfrac{1}{3}x^{1/3} + \dfrac{16}{x} - \dfrac{8}{3}$

E. $y = -\dfrac{2}{3}x + \dfrac{16}{3}$

F. $y = x - 8$

Your answer: B

$$f'(x) = \frac{1}{3}x^{-2/3} - 16(-x^{-2}) = \frac{1}{3(\sqrt[3]{x})^2} + \frac{16}{x^2}$$

$$f'(8) = \frac{1}{3(\sqrt[3]{8})^2} + \frac{16}{8^2} = \frac{1}{3(4)} + \frac{16}{64} = \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = \frac{1}{3}$$

Now $y - 0 = \frac{1}{3}(x-8) \Rightarrow y = \frac{1}{3}x - \frac{8}{3}$

6. (1 point) If $h(x) = \sqrt{e^{2x} + x^3}$ then

A. $h'(x) = e^x + \frac{3}{2}\sqrt{x}$

B. $h'(x) = \dfrac{2e^{2x} + 3x^2}{2(e^{2x} + x^3)}$

C. $h'(x) = \dfrac{2e^{2x} + 3x^2}{2\sqrt{e^{2x} + x^3}}$

D. $h'(x) = \dfrac{1}{2\sqrt{e^{2x} + x^3}}$

E. $h'(x) = \dfrac{1}{2\sqrt{2e^{2x} + 3x^2}}$

F. $h'(x) = \dfrac{2e^{2x} + 3x^2}{2\sqrt{x}}$

$$h'(x) = \frac{1}{2}(e^{2x} + x^3)^{-1/2}((e^{2x})(2) + 3x^2)$$

$$= \frac{2e^{2x} + 3x^2}{2\sqrt{e^{2x} + x^3}}$$

Your answer: C

7. (1 point) Suppose we are given

$$f(x) = \frac{-3 + 6x - x^2}{x}, \quad f'(x) = \frac{-x^2 + 3}{x^2}, \quad f''(x) = -\frac{6}{x^3}.$$

Which one of the following statements is true of the number $x = -\sqrt{3}$?

A. It is ~~not~~ a critical number of f, but f has an inflection point there.

B. f has a local minimum ✓ there, but it is not a global minimum. ✓

C. f has a local ~~maximum~~ there, but it is not a global extremum.

D. f has a local and ~~global~~ minimum there.

E. f has a local ~~maximum~~ and a global minimum there.

F. It is ~~neither~~ a critical number nor an inflection point of f.

$$f'(-\sqrt{3}) = \frac{-(-\sqrt{3})^2 + 3}{(-\sqrt{3})^2} = 0 \quad \therefore -\sqrt{3} \text{ is a critical \#}$$

Your answer: B

$0 = f'(x)$

$\Rightarrow 0 = \dfrac{-x^2 + 3}{x^2}$

$\Rightarrow 0 = -x^2 + 3$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$
$\ominus$	$\oplus$	$\oplus$
$\searrow$	$\nearrow$	$\nearrow$

LOCAL MIN

VA $x=0$

$$\lim_{x \to 0^+} f(x) = -\infty$$

8. (1+3+1=5 points) In this question, you will use the definition of the derivative to find the derivative of a function, and then you will check your answer using the quotient rule.

(a) Give the definition of the derivative of a differentiable function $f(x)$.

Answer: $f'(x) = \lim\limits_{\boxed{h \to 0}} \boxed{\dfrac{f(x+h)-f(x)}{h}}$

(b) Using the definition of the derivative, find the derivative of $f(x) = \dfrac{2x}{3-x}$.

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{2(x+h)}{3-(x+h)} - \dfrac{2x}{3-x}}{h}$$

$$= \lim_{h\to 0} \frac{(2x+2h)(3-x)-2x(3-x-h)}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{6x-2x^2+6h-2xh-6x+2x^2+2xh}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{6\cancel{h}}{\cancel{h}(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{6}{(3-x-h)(3-x)}$$

$$= \frac{6}{(3-x-0)(3-x)}$$

$$= \frac{6}{(3-x)^2}$$

(c) Compute the derivative of $f(x) = \dfrac{2x}{3-x}$ using the quotient rule.

$$f'(x) = \frac{2(3-x)-2x(-1)}{(3-x)^2} = \frac{6-2x+2x}{(3-x)^2} = \frac{6}{(3-x)^2}$$

9. (2 + 2 + 1 = 5 points) A population of snails for a nice restaurant grows at a logistic rate, and is harvested regularly. Denoting the fraction of snails harvested each month by the parameter h, the DTDS governing the growth of this population is given by

$$N_{t+1} = N_t(4.2 - N_t) - hN_t,$$

where t is in months, N_t is the population (in thousands of snails) at time t, and $0 \le h \le 2$.

(a) Find the equilibria N^* of this system, showing your work below. Your answers may be formulas using the parameter h.

$N^* =$ $\boxed{0}$ and $N^* =$ $\boxed{3.2-h}$

$N = f(N)$

$N = N(4.2-N) - hN$

$0 = 4.2N - N^2 - hN - N$

$0 = N(3.2 - N - h)$

$N = 0 \qquad N = 3.2 - h$

If you were unable to answer (a), use the (incorrect) formula $\frac{4}{17}(4-3h)$ for N^* in parts (b) and (c).

(b) The *positive* equilibrium N^* is stable, and the *long-term equilibrium harvest* S is $S = hN^*$. Using your formula in (a), find the value of h that maximizes S, justifying your answer with Calculus.

Optimal value of h: $\boxed{1.6}$

$S(h) = hN^* = h(3.2-h) = 3.2h - h^2$

Now find critical #s of $S(h)$.

$S'(h) = 3.2 - 2h$ Solve $0 = S'(h)$

$0 = 3.2 - 2h$

$2h = 3.2$

$h = 1.6$

$[0, 1.6)$	$(1.6, 2]$
$S'(1) > 0$	$S'(1.9) < 0$
$+$	$-$

since yield increases for $0 \le h < 1.6$ then decreases for $h > 1.6$, it follows that S attains a global maximum when $h = 1.6$

(c) Find the maximum value of the equilibrium harvest S that the restaurant can expect.

Maximum value of S: $\boxed{2.56}$

$S(1.6) = 3.2(1.6) - 1.6^2 = 2.56$

(in thousands of snails)

10. (8 points) We would like to sketch the graph of $y = f(x)$. We have computed for you:

$$f(x) = \frac{x}{x^3-1}, \qquad f'(x) = \frac{-2x^3-1}{(x^3-1)^2} \qquad \text{and} \qquad f''(x) = \frac{6x^2(x^3+2)}{(x^3-1)^3}.$$

Fill in the answers to each of the following limits.

$\lim_{x\to 1^-} f(x) = \boxed{-\infty}$ and $\lim_{x\to 1^+} f(x) = \boxed{\infty}$

$\lim_{x\to -\infty} f(x) = \boxed{0}$ and $\lim_{x\to \infty} f(x) = \boxed{0}$

To find critical #s solve $0=f'(x)$

$0 = \frac{-2x^3-1}{(x^3-1)^2}$

$0=-2x^3-1$

$2x^3=-1$

$x^3=-1/2$ $\Rightarrow x = \sqrt[3]{-0.5} \approx -0.794$

Complete the following tables and then sketch the graph of the function using this information. **Round your answers to three decimal places.**

Critical numbers of f: $\boxed{x \approx -0.794}$

Broken domain (intervals)	$(-\infty,-0.794...)$	$(-0.794..,1)$	$(1,\infty)$
Sign of $f'(x)$ on interval	$+$	$-$	$-$
Behaviour of $y = f(x)$	INC. $\nearrow$	$\searrow$ DEC.	$\searrow$ DEC.

LOCAL MAX @ $(-0.794, 0.529)$

VA $x=1$

To find IP candidates, solve $0=f''(x) \Rightarrow 0 = \frac{6x^2(x^3+2)}{(x^3-1)^3} \Rightarrow 0 = 6x^2(x^3+2)$

$x=0$ $\qquad x=-\sqrt[3]{2} \approx -1.260$

Inflection point candidates: $\boxed{x=0 \text{ and } x \approx -1.260}$

Broken domain (intervals)	$(-\infty,-1.260...)$	$(-1.260..,0)$	$(0,1)$	$(1,\infty)$
Sign of $f''(x)$ on interval	$+$	$-$	$-$	$+$
Behaviour of $y = f(x)$	CONCAVE UP	CONCAVE DOWN	CONCAVE DOWN	CONCAVE UP

On the following page, sketch a graph of the function $y = f(x)$. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**

IP @ $(-1.260, 0.420)$

V.A. $x=1$

On this page, sketch a graph of the function $y = f(x)$ from Question 10. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**