Solution to Midterm Test 1 (A1)

MAT 1322D, Fall 2016
Total = 20 marks

I. Multiple-Choice Questions (3 × 4 = 12 marks)

BBDA

1. The area of the region under the graph of $y = 3 - 2x$ and above the graph of $y = x^2$ is

(A) $\frac{29}{3}$; (B) $\frac{32}{3}$; (C) $\frac{25}{6}$; (D) $\frac{37}{3}$; (E) $\frac{35}{6}$.

Solution. (B) Let $3 - 2x = x^2$. $x^2 + 2x - 3 = 0$. $x = -3, 1$. Since $3 - 2x > x^2$ when $-3 < x < 1$, the area of the region is

$$A = \int_{-3}^{1} ((3-2x) - x^2)dx = \frac{32}{3}.$$

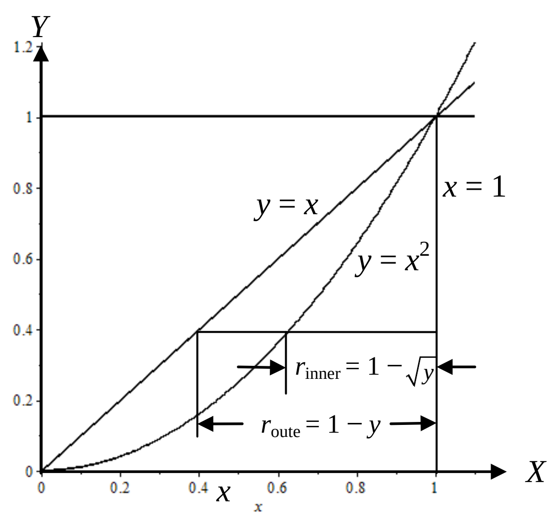
2. Let R be the region under the graph of $y = x$ and above the graph of $y = x^2$. The volume of the solid obtained by revolving R about the line $x = 1$ is given by the integral

(A) $\pi\int_0^1 ((1+y)^2 - (1+\sqrt{y})^2)dy$; (B) $\pi\int_0^1 ((1-y)^2 - (1-\sqrt{y})^2)dy$;

(C) $\pi\int_0^1 ((1-y^2)^2 - (1-y)^2)dy$; (D) $\pi\int_0^1 ((1+\sqrt{y})^2 - (1+y)^2)dy$;

(E) $\pi\int_0^1 ((1-y)^2 - (1-y^2)^2)dy$.

Answer. (B) The picture is as follows:



3. Consider improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$. Which one of the following argument is true?

(A) Since $\frac{1+x}{\sqrt{x}} > \frac{1}{\sqrt{x}}$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ diverges, improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$ diverges.

(B) Since $\frac{1+x}{\sqrt{x}} > \frac{1}{\sqrt{x}}$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$ converges.

(C) Since $\frac{1+x}{\sqrt{x}} < \frac{2}{\sqrt{x}}$, and $\int_0^1 \frac{2}{\sqrt{x}}dx = 2\int_0^1 \frac{1}{\sqrt{x}}dx$ diverges, improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$ diverges.

(D) Since $\frac{1+x}{\sqrt{x}} < \frac{2}{\sqrt{x}}$, and $\int_0^1 \frac{2}{\sqrt{x}}dx = 2\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$ converges.

(E) Since $\frac{1+x}{\sqrt{x}} < \frac{2}{\sqrt{x}}$, and $\int_0^1 \frac{2}{\sqrt{x}}dx = 2\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, improper integral $\int_0^1 \frac{1+x}{\sqrt{x}}dx$ diverges.

Answer. (D)

4. Suppose that a spring of length 1 meter needs 10 Newton to hold at a length 1.1 meters. The work, in Joules, needed to stretch the spring from 1 meter to 1.5 meters is

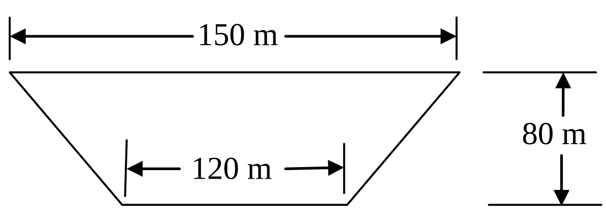
(A) 12.5; (B) 25; (C) 30; (D) 37.5; (E) 50.

Solution. (A) Let the extra length of the string be x. Then the force needed to stretch the spring from extra length x meters to $x + dx$ meters is $\frac{10}{0.1}x\,dx = 100x\,dx$. The total work is

$100\int_0^{0.5} x\,dx = 100 \times 0.125 = 12.5$ Joule.

II. Detailed Answer Questions (4 × 2 = 8 marks)

5. Suppose a dam has the shape of a trapezoid as shown in the following figure.



The water level is 5 meters under the top of the dam. Construct, but not evaluate, an integral that calculates the force, in Newtons, acting on the dam. Assume the density of water is ρ kg/m^3, and the acceleration of gravity is g m/sec^2.

Solution. Look at a horizontal stripe of the dam x meters above the bottom with height dx. The area of this stripe is $A(x) = \left(\frac{3}{8}x+120\right)dx$. The depth of this tripe is $D(x) = 75 - x$. The pressure is $P(x) = \rho g D(x) = \rho g(75 - x)$. The force acting on this stripe is $dF = \rho g\left(\frac{3}{8}x+120\right)(75-x)dx$.

The total force is

$$F = \rho g\int_0^{75}\left(\frac{3}{8}x+120\right)(75-x)dx.$$

Alternative solutions:

A. If you let x be the distance between a stripe of the dam and the top of the dam, than the integral is

$$F = \rho g\int_5^{80}\left(\frac{3}{8}(80-x)+120\right)(x-5)dx.$$

B. If you let x be the distance between a stripe of the dam and the water surface, then the integral is

$$F = \rho g\int_0^{75}\left(\frac{3}{8}(75-x)+120\right)x\,dx.$$

6. Let R be the region between the graph of $y = e^x - 1$ and the x-axis, $0 \le x \le 1$. Assuming it has a uniform density $\rho = 1$. Find the moments of R respect to x-axis and y-axis, and coordinates of the center of mass of this region.

Solution. The moments:

$$M_x = \frac{1}{2}\int_0^1 (e^x-1)^2 dx = \frac{1}{2}\int_0^1 (e^{2x}-2e^x+1)dx = \frac{1}{2}\left[\frac{1}{2}e^{2x}-2e^x+x\right]_{x=0}^1 = \frac{1}{2}\left(\frac{1}{2}(e^2-1)-2(e-1)+1\right)$$
$$= \frac{1}{4}(e^2-4e+5),$$

$$M_y = \int_0^1 x(e^x-1)dx = \left[x(e^x-x)\right]_{x=0}^1 - \int_0^1 (e^x-x)dx = e-1-\left[e^x-\frac{1}{2}x^2\right]_{x=0}^1 = \frac{1}{2}.$$

The mass of *R* equals its area: $m = A = \int_0^1 (e^x - 1)dx = e - 2$. Hence, $\overline{x} = \frac{M_y}{m} = \frac{1}{2(e-2)}$,

$\overline{y} = \frac{M_x}{m} = \frac{e^2 - 4e + 5}{4(e-2)}$.