MAT 1332, Winter 2015, Assignment 1
Due Wednesday January 27 in the math department dropboxes by 7:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

Instructor (circle one): Robert Smith? Petko Kitanov Catalin Rada

DGD (circle one): 1 2 3 4

Student Name ______________________ Student Number ______________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________

QUESTION 1. Calculate the following

(a) $\displaystyle\int \frac{x^{3/2} - 5x}{\sqrt{x}} dx$

$$\begin{aligned} I &= \int \left(\frac{x^{3/2}}{\sqrt{x}} - \frac{5x}{\sqrt{x}} \right) dx \\ &= \int x\,dx - 5 \int x^{1/2}\,dx \\ &= \frac{x^2}{2} - 5\frac{x^{3/2}}{3/2} + C \\ &= \frac{x^2}{2} - \frac{10}{3}x^{3/2} + C \end{aligned}$$

(b) $\displaystyle\int \left(\sin x + 4x^2 - \frac{6x}{\sqrt[3]{x}} \right) dx$

$$\begin{aligned} I &= \int \sin x\,dx + 4\int x^2\,dx - 6\int x^{2/3}\,dx \\ &= -\cos x + 4\frac{x^3}{3} - 6\frac{x^{5/3}}{5/3} + C \\ &= -\cos x + \frac{4}{3}x^3 - \frac{18}{5}x^{5/3} + C \end{aligned}$$

(c) $\int \sec^2 x \tan x dx$

Setting $u = \tan x$, we have $du = \sec^2 x\, dx$ and hence by substitution

$$\begin{aligned} I &= \int u\, du \\ &= \frac{u^2}{2} + C \\ &= \frac{(\tan x)^2}{2} + C \end{aligned}$$

Alternatively, we could use the substitution $u = \cos x$ with $du = -\sin x dx$ and write

$$I = \int \frac{\sin x}{\cos^3 x} dx = -\int u^{-3} du = \frac{1}{2u^2} + k = \frac{1}{2\cos^2 x} + k$$

Note that these are in fact the same answer, only with different constants, because wecan write

$$\frac{1}{2}\tan^2 x + C = \frac{\sin^2 x}{2\cos^2 x} + C = \frac{1 - \cos^2 x}{2\cos^2 x} + C = \frac{1}{2\cos^2 x} - \frac{1}{2} + C = \frac{1}{2\cos^2 x} + k$$

absorbing the $-\frac{1}{2}$ into the constant of integration.

(d) $\int (x^3 + x)^{10}(3x^2 + 1) dx$

Setting $u = x^3 + x$, we have $du = (3x^2 + 1)\, dx$ and by substitution

$$\begin{aligned} I &= \int u^{10}\, du \\ &= \frac{u^{11}}{11} + C \\ &= \frac{(x^3 + x)^{11}}{11} + C \end{aligned}$$

(e) $\int x^4 \ln x dx$

[2] Using integration by parts, we have

$$u = \ln x \qquad v = x^4$$

$$u' = \frac{1}{x} \qquad v = \frac{x^5}{5}$$

Then the integral is

$$\begin{aligned} I &= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4\, dx \\ &= \frac{x^5}{5} \ln x - \frac{x^5}{25} + C \end{aligned}$$

(f) $\int x \sin(x/2) dx$

Using integration by parts, we have

$$u = x \qquad v = \sin\left(\frac{x}{2}\right)$$
$$u' = 1 \qquad v = -2\cos\left(\frac{x}{2}\right)$$

Then the integral is

$$\begin{aligned} I &= -2x\cos\left(\frac{x}{2}\right) + 2\int \cos\left(\frac{x}{2}\right)\, dx \\ &= -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C \end{aligned}$$

(g) $\int \frac{e^{1/x}}{5x^2} dx$

[2] Letting $u = \frac{1}{x}$, we have $du = -\frac{1}{x^2}\, dx$ and by substitution

$$\begin{aligned} I &= -\frac{1}{5}\int e^u\, du \\ &= -\frac{e^u}{5} + C \\ &= -\frac{e^{1/x}}{5} + C \end{aligned}$$

QUESTION 2. Consider the function $f(x) = \dfrac{9x^2 - 17x + 6}{x(x-1)(x-2)}$

(a) Decomposer $f(x)$ into partial fractions.

We can write

$$\frac{9x^2 - 17x + 6}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

Multiplying throughout, we have

$$9x^2 - 17x + 6 = A(x-1)(x-2) + Bx((x-2) + Cx(x-1)$$

Choosing key values of x, we have

$$\begin{array}{lcr} x = 0 & 6 = 2A & A = 3 \\ x = 1 & -2 = -B & B = 2 \\ x = 2 & 8 = 2C & C = 4 \end{array}$$

Hence we have

$$\frac{9x^2 - 17x + 6}{x(x-1)(x-2)} = \frac{3}{x} + \frac{2}{x-1} + \frac{4}{x-2}$$

(b) Calculate $\int f(x)dx$

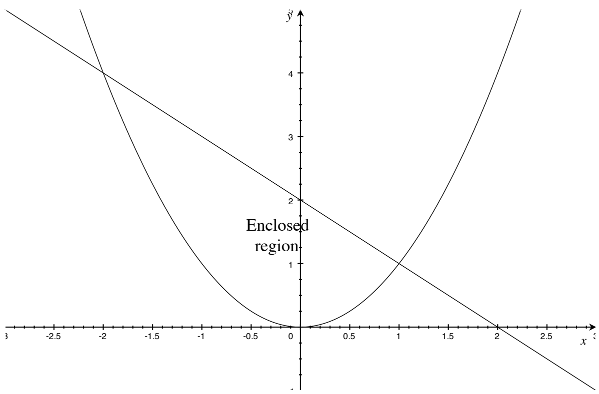
$$I = \int f(x)\,dx = 3\ln|x| + 2\ln|x-1| + 4\ln|x-2| + C$$

Question 3.

(a) Sketch the region enclosed by the curves $y = 2 - x$ and $y = x^2$.

(b) Find the area of this region.

[1] (a) Note that there is only one region enclosed by both curves. (All others are unbounded or require another enclosing curve.)



[2] (b) First we need to find the pointsof intersection. Equating the two curves, we have

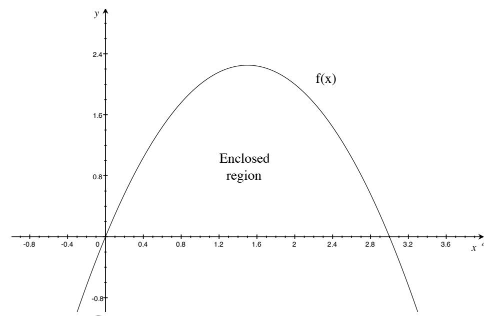
$$\begin{aligned} x^2 &= 2 - x \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x &= 1, -2. \end{aligned}$$

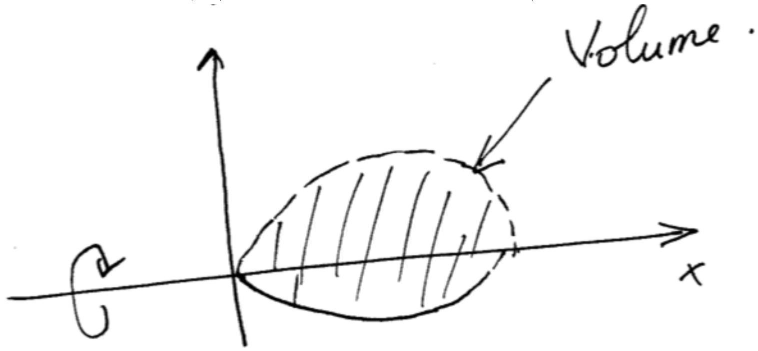
Notice that the line is above the quadratic for the region we are interested in. Then the area is

$$\begin{aligned} \int_{-2}^{1} (2 - x - x^2)dx &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} \\ &= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\ &= \frac{7}{6} + \frac{10}{3} \\ &= \frac{9}{2} \text{ units}^2 \end{aligned}$$

QUESTION 4. Consider the region enclosed by the curve $y = 3x - x^2$ and the x-axis. Determine the volume of revolution described by rotating the surface around the x-axis. Sketch the two-dimensional region and the rotated volume on separate graphs.

[1]





[2] The x-intercepts are $x = 0$ and $x = 3$. We thus have

$$\begin{aligned} V &= \int_0^3 \pi (f(x))^2 \, dx \\ &= \pi \int_0^3 (3x - x^2)^2 \, dx \\ &= \pi \int_0^3 (9x^2 - 6x^3 + x^4) \, dx \\ &= \pi \left[\frac{9x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5} \right]_0^3 \\ &= \frac{81}{10}\pi \quad \text{units}^3 \end{aligned}$$

Question 5. Determine whether each of the following integrals converges or diverges. If it converges, calculate the value.

(a) $\displaystyle\int_1^3 \frac{x}{(x^2-9)^{4/3}}dx$

Set $u = x^2 - 9$ so that $\frac{du}{dx} = 2x$ and hence $dx = \frac{du}{2x}$. Then the integral is

$$\begin{aligned} I &= \lim_{c\to 3^-}\int_1^c \frac{x}{(x^2-9)^{4/3}}dx = \lim_{c\to 3^-}\int_{x=1}^{x=c}\frac{x}{u^{4/3}}\frac{du}{2x} = \lim_{c\to 3^-}\int_{x=1}^{x=c} u^{-4/3}du \\ &= \lim_{c\to 3^-} -\frac{3}{2}u^{-1/3}\Big|_{x=1}^{x=c} = \lim_{c\to 3^-} -\frac{3}{2}(x^2-9)^{-1/3}\Big|_1^c \\ &= \lim_{c\to 3^-}\left[-\frac{3}{2}\frac{1}{(c^2-9)^{1/3}} + \frac{3}{2}\frac{1}{(-8)^{1/3}}\right] = \infty \end{aligned}$$

Hence the integral diverges.

(b) $\displaystyle\int_{-1}^1 \frac{x^3}{\sqrt{1-x^4}}dx$

[2] First we have

$$\begin{aligned} I &= \int_{-1}^1 \frac{x^3}{\sqrt{1-x^4}}\,dx \\ &= \int_{-1}^0 \frac{x^3}{\sqrt{1-x^4}}\,dx + \int_0^1 \frac{x^3}{\sqrt{1-x^4}}\,dx \\ &= \lim_{c\to -1^+}\int_c^0 \frac{x^3}{\sqrt{1-x^4}}\,dx + \lim_{d\to 1^-}\int_0^d \frac{x^3}{\sqrt{1-x^4}}\,dx \end{aligned}$$

Set $u = 1 - x^4$ so that $\frac{du}{dx} = -4x^3$ and $dx = \frac{du}{-4x^3}$. Then by substitution, we have

$$= \lim_{c\to -1^+}[F(0) - F(c)] + \lim_{d\to 1^-}[F(d) - F(0)]$$

where

$$F(x) = \int \frac{x^3}{\sqrt{1-x^4}}\,dx = \int \frac{x^3}{\sqrt{u}}\frac{du}{-4x^3} = -\frac{1}{4}\int u^{-1/2} = -\frac{1}{4}u^{1/2} + C = -\frac{1}{4}\sqrt{1-x^4} + C$$

Then

$$I = \lim_{c\to -1^+}\left[-\frac{1}{4} + \frac{1}{4}\sqrt{1-c^4}\right] + \lim_{d\to 1^-}\left[-\frac{1}{4}\sqrt{1-d^4} + \frac{1}{4}\right] = 0$$

Hence the integral converges.

QUESTION 6. The growth of a raccoon can be characterised by $\frac{dw}{dt} = \frac{1}{\ln(t+2)}$ where t is the time in months and w is the weight in kilograms. Use five Riemann sums to estimate the amount that the raccoon grows between the first and third month.

[3] We need to calculate

$$\int_1^3 \frac{1}{\ln(t+2)}\,dt$$

Note that this integral cannot be solved using the traditional methods.

If we have five Riemann sums, then the width of each rectangle is $\Delta t = \frac{3-1}{5} = \frac{2}{5}$. The integral is then approximated by

$$S_D = \Delta t\,[f(t_1) + f(t_2) + f(t_3) + f(t_4) + f(t_5)]$$

where

$$t_1 = 1, \quad t_2 = 1.4, \quad t_3 = 1.8, \quad t_4 = 2.2 \quad \text{and} \quad t_5 = 2.6.$$

are the t-values of the five Riemann rectangles from the left. Then the sum is

$$S = 0.4\left[\frac{1}{\ln(3)} + \frac{1}{\ln(3.4)} + \frac{1}{\ln(3.8)} + \frac{1}{\ln(4.2)} + \frac{1}{\ln(4.6)}\right] \approx 1.531421 \text{ kg}$$

(Note that you lose half a mark if you don't specify the units.)

However, the question didn't specify whether we should have the left or right Riemann sums, so either is acceptable. Taking the right sums, we have

$$S = 0.4\left[\frac{1}{\ln(3.4)} + \frac{1}{\ln(3.8)} + \frac{1}{\ln(4.2)} + \frac{1}{\ln(4.6)} + \frac{1}{\ln(5)}\right] \approx 1.415859 \text{ kg}$$