MATH2004A – Test 3 – 4:35 pm - 5:25 pm, Nov 5

Name: Student Number:

Total: 20 marks
Closed book, no calculator!

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1. [3 points] Given the linear equation of the plane Π: $2x + 3y + 4z - 4 = 0$ and the line $L : \mathbf{r} =< 1, 2, 3 > +t < 3, 2, -1 >$, find the intersection between the plane Π and the line L.

Solution: The parametric equation of the line is:

$$x = 1 + 3t, y = 2 + 2t, z = 3 - t.$$

Substitute this into the plane we get

$$2(1 + 3t) + 3(2 + 2t) + 4(3 - t) - 4 = 0, \Rightarrow t = -2.$$

Thus the point is (-5,-2,5).

2. [5 points] (i) (3 points) Find the linear equation of the plane Π passing through the three points $P(1, 2, -1)$, $Q(3, 1, -1)$ and $R(1, 1, 0)$.

(ii) (2 points) Find the distance from the point $S(8, 1, 1)$ to the plane Π in (i).

Solution: (i) We have

$$\vec{PQ} =< 2, -1, 0 >, \quad \vec{PR} =< 0, -1, 1 > .$$

The normal vector of the plane is

$$\mathbf{n} = \vec{PQ} \times \vec{PR} =< -1, -2, -2 > .$$

The equation of the plane is

$$< -1, -2, -2 > \cdot (\mathbf{r} - < 1, 2, -1 >) = 0$$

i.e.,

$$-x - 2y - 2z + 3 = 0, or, x + 2y + 2z - 3 = 0.$$

(ii). The distance is:

$$d = \frac{|8 + 2(1) + 2(1) - 3|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

3. [2 points] Let $\mathbf{r}(t) =< 6t, 8t, te^t >= 6t\mathbf{i} + 8t\mathbf{j} + te^t\mathbf{k}$. Calculate $\int_0^1 \mathbf{r}(t)dt$.
Solution:

$$\int_0^1 \mathbf{r}(t)dt =< \int_0^1 6tdt, \int_0^1 8tdt, \int_0^1 te^t dt >=< 3, 4, e > .$$

4. [5 points] Let C be $\mathbf{r}(t) =< 3t, \cos(4t), \sin(4t) >= 3t\mathbf{i} + \cos(4t)\mathbf{j} + \sin(4t)\mathbf{k}$.
(i)(2 points) Find the length of the curve from $t = 0$ to $t = 2$.
(ii) (3 points) Find the unit tangent vector $T(t)$ and the unit normal vector $N(t)$.
Solution: (i) $\mathbf{r}'(t) =< 3, -4\sin(4t), 4\cos(4t) >$, which gives $|\mathbf{r}'(t)| = 5$.
So the length is

$$L = \int_0^2 |\mathbf{r}'(t)|dt = \int_0^2 5dt = 10.$$

(ii) From (i) we see that $\mathbf{r}'(0) =< 3, 0, 4 >$, $|\mathbf{r}'(0)| = 5$. Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} =< 3/5, -4/5\sin(4t), 4/5\cos(4t) > .$$

$$\mathbf{T}'(t) =< 0, -16/5\cos(4t), -16/5\sin(4t) >, \quad |\mathbf{T}'(t)| = 16/5.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} =< 0, -\cos(4t), -\sin(4t) > .$$

5. [5 points] Let $\mathbf{r}(t) =< t - 1, t^2, e^{2t} >= (t-1)\mathbf{i} + t^2\mathbf{j} + e^{2t}\mathbf{k}$.

(i) (2 points) Find the acceleration.

(ii) (3 points) Find the curvature $\kappa(0)$ of $\mathbf{r}(t)$ at $t = 0$.

Solution: i) $\mathbf{r}'(t) =< 1, 2t, 2e^{2t} >$. The acceleration is $\mathbf{a}(t) = \mathbf{r}''(t) =< 0, 2, 4e^{2t} >$.

(ii) At $t = 0$, $\mathbf{r}'(0) =< 1, 0, 2 >$, $\mathbf{r}''(0) =< 0, 2, 4 >$, $\mathbf{r}'(0) \times \mathbf{r}''(0) =< -4, -4, 2 >$. Hence

$|\mathbf{r}'(0)| = \sqrt{5}$, $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = | < -4, -4, 2 > | = 6$.

The curvature at $t = 0$ is:

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{6}{\sqrt{5}^3}.$$