

1. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x \geq 0, y \geq 0 \text{ and } z \geq 0\}$. Which one of the following statements is true?

A. $(0, 0, 0) \in X$ and X is closed under multiplication by scalars. F

B. $(0, 0, 0) \notin X$ but X is closed under addition. F $(0, 0, 0) \in X$

C. X is closed under addition but X is not closed under multiplication by scalars. T

D. X is closed under addition and X is closed under multiplication by scalars. F

E. X is not closed under addition but X is closed under multiplication by scalars. X F

F. None of the other statements is true.

X is not closed under multⁿ by scalars: $v = (1, 0, 0) \in X$ but
 $-1 \cdot v = (-1, 0, 0) \notin X$.

X is closed under addⁿ ("tve + tve is tve")

2. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent if $a = b = c = 0$ implies $au + bv + cw = 0$. F

II. A set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ implies $a = b = c = 0$. (def'n) T

III. A set $\{u, v, w\} \subset V$ of vectors spans a vector space V if every vector in V is a linear combination of u and w . T

IV. $\{(1, -1), (1, 1)\}$ is an orthogonal set in \mathbf{R}^2 . T

A. Only I & II

B. Only II & IV

C. Only II & III

D. Only I & III & IV

E. Only II & III & IV

F. All of the above statements are true.

3. In a linear system $Ax = b$, with n equations and n unknowns, the rank of A is $n - 1$ and the rank of the augmented matrix $[A | b]$ is also $n - 1$. Which one of the following statements is true?

- A. The system has no solution. F
- B. The system has a unique solution. F $\text{rank } A = n - 1 < n = \# \text{ variables}$
- C. The system has infinitely many solutions. T $\text{rank } A = \text{rank } A|b < \# \text{ variables}$
- D. The system has exactly $n - 1$ solutions.
- E. The determinant of A is non-zero.
- F. Such a system cannot exist.

4. If C is an $m \times 2$ matrix and $D = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$, then the second column of the matrix CD is

- A. not defined unless $m = 2$.
- B. twice the first column of C .
- C. the same as the first column of C .
- D. the same as the second column of C .
- E. the sum of the first and the second columns of C .
- F. the sum of twice the first column of C and three times the second column of C .

$$C = [C_1 \ C_2]$$

$$CD = [C_1 \ C_2] \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= [C_2 \ 2C_1 + 3C_2 \ C_1 + C_2]$$

5. The dimension of $S = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid A = A^t\}$ is:

- A. 0
- B. 1
- C. 2
- D. 3**
- E. 4
- F. 5

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{M_3} \right\}$$

Moreover, $\{M_1, M_2, M_3\}$ is l.i., so $\dim S = 3$

6. Let A be a 7×12 matrix such that $\text{rank } A = 6$. As usual, $\ker A = \{x \in \mathbb{R}^{12} \mid Ax = 0\}$ and $\text{col } A = \{Ax \mid x \in \mathbb{R}^{12}\}$. Which of the following statements is true?

- A. $\dim \text{col } A = 6, \dim \ker A = 1$
- B. $\text{col } A = \mathbb{R}^{12}, \dim \ker A = 6$
- C. $\text{col } A = \mathbb{R}^7, \dim \ker A = 5$
- D. $\dim \text{col } A = 5, \ker A = \mathbb{R}^{12}$
- E. $\dim \text{col } A = 6, \dim \ker A = 6$**
- F. $\text{col } A = \{0\}, \dim \ker A = 5$

$$7 \begin{array}{|c|c|} \hline & 12 \\ \hline A & 0 \\ \hline \end{array}$$

$$\text{rank } A = 6.$$

$$\therefore \dim \ker A = 12 - 6 = 6.$$

$$\dim \text{col } A = \text{rank } A = 6$$

7. If two $n \times n$ matrices A and B satisfy $A^t = B^{-1}$ and $B^t = -B^{-1}$ then $(ABA)^t$ is always

A. $-B^3$

B. B^2A

C. $-B^{-3}$

D. B^{-3}

E. B^3

F. AB^2

$$\begin{aligned} (ABA)^t &= A^t B^t A^t \\ &= B^{-1} (-B^{-1}) B^{-1} \\ &= -B^{-3} \end{aligned}$$

8. Which two of the following statements are false?

- (i) For all invertible $n \times n$ matrices A and B , $\det(A^{-1}BA) = \det B$ T
- (ii) For all invertible $n \times n$ matrices A and B , $\det(A^{-1}B^{-1}AB) = 1$ T
- (iii) For all $n \times n$ matrices A and B , $(A^t B^t)^t = AB$ F $(AB^t)^t = BA$
- (iv) For all invertible $n \times n$ matrices A and B , $(ABA^{-1})^{-1} = A^{-1}B^{-1}A$ F $(ABA^{-1})^{-1} = AB^{-1}A^{-1}$
- (v) For all $n \times n$ matrices A and B , $\det(A^t B) = \det(B^t A)$ T

A. (i) and (iii)

B. (ii) and (iii)

C. (iii) and (iv)

D. (ii) and (iv)

E. (ii) and (v)

F. (i) and (v)

9. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix} = 4 \begin{vmatrix} g & a & d-2a \\ h & b & e-2b \\ i & c & f-2c \end{vmatrix}$

A. 24
 B. -24
 C. 12
 D. -12
 E. 6
 F. -6

$\left. \begin{matrix} \\ \\ \end{matrix} \right\} 2C_2 + C_3 \rightarrow C_3$

$= 4 \begin{vmatrix} g & a & d \\ h & b & e \\ i & c & f \end{vmatrix}$

$\left. \begin{matrix} \\ \\ \end{matrix} \right\} (\det A = \det A^t)$

$= 4 \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

$\left. \begin{matrix} \\ \\ \end{matrix} \right\} R_1 \leftrightarrow R_1$

$= -4 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

$\left. \begin{matrix} \\ \\ \end{matrix} \right\} R_2 \leftrightarrow R_3$

$= +4 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 12$

10. Which of the following are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f: \mathbf{R} \rightarrow \mathbf{R}\}$?

$$S = \{\cos x, \sin x\}$$

$$T = \{1, \cos^2 x, \sin^2 x\}$$

$$U = \{1, 2 \cos^2 x, 3 \sin^2 x\}$$

$$V = \{1, \cos x, \sin x\}$$

We know $\sin^2 x + \cos^2 x = 1, \forall x$,

So T and U are dependent.

Hence the answer must be D

A. T and V .

B. T and U .

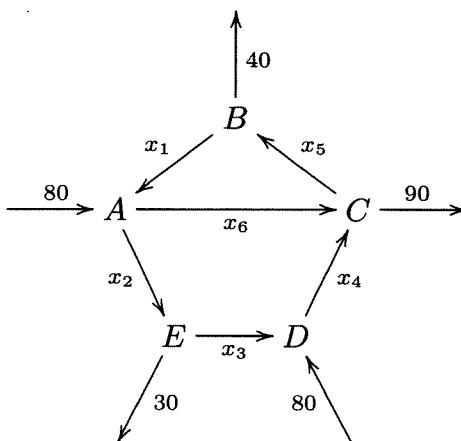
C. S and T .

D. S and V .

E. S, U and V .

F. S, U and T .

11. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the traffic flow, together with all the constraints on the variables $x_i, i = 1, \dots, 6$.

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	$80 + x_1$	=	$x_2 + x_6$
B	x_5	=	$x_1 + 40$
C	$x_4 + x_6$	=	$x_5 + 90$
D	$80 + x_3$	=	x_4
E	x_2	=	$30 + x_3$

Constraints.

$$x_i \geq 0, \quad i = 1, \dots, 6 \quad (\text{one-way streets})$$

$$x_i \in \mathbb{Z} \quad (\text{\# of cars.})$$

11(b). The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \Delta & t & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 40 \\ 0 & 0 & 1 & 0 & -1 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 1 & 90 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= -40 + \Delta \\ x_2 &= 40 + \Delta - t \\ x_3 &= 10 + \Delta - t \\ x_4 &= 90 + \Delta - t \\ x_5 &= \Delta \\ x_6 &= t \end{aligned} \quad ; \quad \Delta, t \in \mathbb{R}$$

c) If \overline{AC} were closed due to roadwork, find the minimum flow along \overline{ED} , using your results from (b).

(You must justify all your answers.)

If \overline{AC} is closed, $x_6 = t = 0$. We look for the minimum of x_3 .
Note $x_3 = 10 + \Delta$, so we seek the minimum of Δ , first.

Constraints.

$$\left. \begin{aligned} x_1 \geq 0 &\Leftrightarrow \Delta \geq 40 \\ x_2 \geq 0 &\Leftrightarrow \Delta \geq -40 \\ x_3 \geq 0 &\Leftrightarrow \Delta \geq -10 \\ x_4 \geq 0 &\Leftrightarrow \Delta \geq -90 \\ x_5 \geq 0 &\Leftrightarrow \Delta \geq 0 \end{aligned} \right\} \Rightarrow \Delta \geq 40$$

$$\therefore x_3 = 10 + \Delta \geq \underline{50}$$

$$\frac{1}{2}$$

$\frac{1}{2}$ - just.

12. Let $v_1 = (1, 0, 0, -1)$, $v_2 = (1, -1, 0, 0)$, $v_3 = (1, 0, 1, 0)$ and set $U = \text{span}\{v_1, v_2, v_3\} \subset \mathbb{R}^4$.

$$\{(x, y, z, w) \mid x + y - z + w = 0\}$$

1.5 a) Briefly explain (you may refer to results learned in class or from the book) why U is a subspace of \mathbb{R}^4 , and why $\{v_1, v_2, v_3\}$ is a basis for U .

$$x + y + w = z$$

2 b) Use the **Gram-Schmidt algorithm** to find an orthogonal basis for U .

2.5 c) Find the best approximation by a vector in U to the vector $(2, 0, 2, 4)$.

a) U is a subspace because it is the span of v_1, v_2 & v_3 . $\frac{1}{2}$

$\{v_1, v_2, v_3\}$ is a basis for U since it spans U (by defn) and

is l.o.i. $\frac{1}{2}$ because $\text{rank} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 3$. $\frac{1}{2}$

So U (being the row space of \uparrow) has dimension 3.

b) We apply Gram Schmidt: 1 correct

Set $u_1 = v_1$

$$\tilde{u}_2 = v_2 - \frac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = (1, -1, 0, 0) - \frac{1}{2} (1, 0, 0, -1) = \left(\frac{1}{2}, -1, 0, \frac{1}{2}\right)$$

We set $u_2 = (1, -2, 0, 1)$ to avoid fractions.

Now $\tilde{u}_3 = v_3 - \frac{v_3 \cdot u_1}{\|u_1\|^2} u_1 - \frac{v_3 \cdot u_2}{\|u_2\|^2} u_2$

$\frac{1}{2}$ - vectors are \perp

$\frac{1}{2}$ - vectors $\in U$

$$= v_3 - \frac{1}{2} (1, 0, 0, -1) - \frac{1}{6} (1, -2, 0, 1)$$

$$= (1, 0, 1, 0) + \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) + \left(-\frac{1}{6}, \frac{1}{3}, 0, -\frac{1}{6}\right)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, 1, \frac{1}{3}\right)$$

So set $u_3 = (1, 1, 3, 1)$. Then $\{u_1, u_2, u_3\}$ is an orthogonal basis for U .

1- correct formula

c) $\text{proj}_U (2, 0, 2, 4) = \frac{(2, 0, 2, 4) \cdot (1, 0, 0, -1)}{\|u_1\|^2} (1, 0, 0, -1) + \frac{(2, 0, 2, 4) \cdot (1, -2, 0, 1)}{\|u_2\|^2} (1, -2, 0, 1) + \frac{(2, 0, 2, 4) \cdot (1, 1, 3, 1)}{\|u_3\|^2} (1, 1, 3, 1)$

$\frac{1}{2}$

$$= -u_1 + u_2 + u_3 = (1, -1, 3, 3) \frac{1}{2}$$

$\frac{1}{2}$ ans $\in U$

13. Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$.

- a) Find the characteristic polynomial of A , and use this to show that the eigenvalues of A are 5 and -1 .
 b) Find a basis of $E_5 = \{v \in \mathbb{R}^3 \mid Av = 5v\}$.
 c) Find a basis of $E_{-1} = \{v \in \mathbb{R}^3 \mid Av = -v\}$.
 d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If this is not possible, explain why.

a) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} \stackrel{\text{row 2}}{=} -(1+\lambda) \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = -(1+\lambda) \{(2-\lambda)^2 - 9\}$
 $= -(1+\lambda) \{\lambda^2 - 4\lambda - 5\} = -(1+\lambda)(\lambda+1)(\lambda-5) = -(1+\lambda)^2(\lambda-5)$. Hence the
 evals of A are -1 and 5 .

b) $E_5 = \ker \begin{bmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 3 & 0 & -3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(s, 0, s) \mid s \in \mathbb{R}\}$. Hence
 $\{(1, 0, 1)\}^{v_1}$ is a basis for E_5 .

c) $E_{-1} = \ker \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-t, s, t)\}$. Hence
 $\{(-1, 0, 1)\}^{v_2}, \{(0, 1, 0)\}^{v_3}$ is a basis for E_{-1} .

d) Since $\dim E_{-1} + \dim E_5 = 3$, A is diagonalizable. Set

$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
 block col. form

Then $P^{-1}AP = D$. (P is inv. since $\{v_1, v_2, v_3\}$ is an orthogonal & hence l.i. set!)

14. Let $u = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and define a linear transformation $S: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ by

$$S(v) = u \times v, \quad \text{for all } v \in \mathbf{R}^3,$$

where " $u \times v$ " denotes cross product of u and v . (You do not have to prove that S is linear.)

a) If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$, show that $S(v) = \begin{bmatrix} -2y + z \\ 2x - 2z \\ -x + 2y \end{bmatrix}$.

b) Find a 3×3 matrix A such that $S(v) = Av$, where Av denotes the matrix product of A and v .

c) Find a basis for $\ker S = \{v \mid S(v) = 0\}$, and give a complete geometric description of $\ker S$.

d) Find a basis for $\text{im } S = \{S(v) \mid v \in \mathbf{R}^3\}$, and give a complete geometric description of $\text{im } S$.

a) $S(v) = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ x & y & z \end{vmatrix} = (z - 2y, -(2z - 2x), 2y - x)$, as req'd.

b) If $A = [S e_1 \ S e_2 \ S e_3] = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$, then $Sv = Av$.

c) We know $\ker S = \ker A = \ker \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{span} \left\{ \left(1, \frac{1}{2}, 1 \right) \right\}$, so $\{ (2, 1, 2) \}$ is a basis for $\ker S$.

This is the line through 0 with direction $(2, 1, 2)$.

d) We know $\text{im } S = \text{col } A = \text{span} \left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \right\}$ from previous work. This is the plane through 0 with normal $(2, 1, 2)$

(from geometric means, or by computing $n = v_1 \times v_2$).

15. a) State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case*.

i) Suppose C is an invertible 3×3 matrix, and that $\{v_1, v_2, v_3\}$ is linearly independent ^{in \mathbb{R}^3} . Then $\{Cv_1, Cv_2, Cv_3\}$ is also linearly independent.

Let $A = [v_1 \ v_2 \ v_3]$. Then A is invertible, so
 $(v_j = j^{\text{th}} \text{ col of } A)$

CA is too. But $CA = C[v_1 \ v_2 \ v_3] = [Cv_1 \ Cv_2 \ Cv_3]$.

So the cols of CA are l.i.o.

ANSWER

TRUE

ii) $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.

The eigenvalues of $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ are 3 and 0 (since

$\begin{vmatrix} 3-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda(\lambda-3)$, which are distinct. Hence

$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ is diag'ble.

ANSWER

TRUE

15 a) (cont.)

iii) If the columns of a 17×6 matrix A form an orthogonal set in \mathbf{R}^{17} , then $\text{rank } A = 6$.

$$A = [c_1 \ c_2 \ \dots \ c_6]$$
 Since $\{c_1, \dots, c_6\}$ is orthogonal, $\{c_1, c_2, \dots, c_6\}$ is l.i. Hence the cols of A are l.i. so $\text{rank } A = 6$ ($= \# \text{ cols}$)

ANSWER

TRUE

15 b). Let A be a $n \times n$ matrix with real entries. Give three additional statements equivalent to

“ A is ; invertible”

one each in terms of

(i) the homogeneous linear system $Ax = 0$:

$$Ax = 0 \Rightarrow x = 0$$

(ii) the rank of A :

$$\text{rank } A = n$$

(iii) the determinant of A :

$$\det A \neq 0$$

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

In parts (a) and (b), A denotes a symmetric 100×100 matrix.

a) Prove that $(Au) \cdot u' = u \cdot (Au')$ for all $u, u' \in \mathbf{R}^{100}$, where “ \cdot ” denotes the dot product.

$$(Au) \cdot u' = (Au)^t u' = u^t A^t u' = u^t A u' = u \cdot (Au').$$

\uparrow dot \uparrow matrix product

Now let $v_\lambda \in \mathbf{R}^{100}$ be an eigenvector of A with eigenvalue λ , and set $W = \{w \in \mathbf{R}^{100} \mid w \cdot v_\lambda = 0\}$.

b) Prove that if $w \in W$, then $Aw \in W$. Let $w \in W$.

$$\begin{aligned} \text{By } A, \quad (Aw) \cdot v_\lambda &= w \cdot (Av_\lambda) = w \cdot (\lambda v_\lambda) = \lambda (w \cdot v_\lambda) \\ &= \lambda \cdot 0 \\ &= 0. \end{aligned}$$

Hence $(Aw) \cdot v_\lambda = 0$, i.e. $Aw \in W$.