School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2016

SOLUTIONS TO TEST 1

Only calculators are permitted, 1 or more blank sheets permitted for roughs

Print Name : _____

Student Number: _____

Tutorial Section (A1, A4, ...): _____

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** Which of the following vectors is orthogonal to the two vectors $(4,0,2)$ and $(0,6,10)$.

   (a) $2\mathbf{i}-10\mathbf{j}+\mathbf{k}$, (b) $-3\mathbf{i}+5\mathbf{j}-6\mathbf{k}$, (c) $\mathbf{i}+10\mathbf{j}+2\mathbf{k}$, $\boxed{(d)}$ $-3\mathbf{i}-10\mathbf{j}+6\mathbf{k}$,
   (e) None of these

2. **[2 marks]** Find the angle between the planes $\Pi_1: \quad x+y+z=0$ and $\Pi_2: \quad z=0$.

   (a) $\text{Arccos}\,(1/3)$, (b) $\text{Arccos}\,(1/\sqrt{2})$, $\boxed{(c)}$ $\text{Arccos}\,(1/\sqrt{3})$, (d) $\text{Arccos}\,(2/\sqrt{3})$. (e) None of these.

3. **[2 marks]** Find the normal vector of the plane Π through the three points $P(2,1,0)$, $Q(1,0,1)$ and $R(0,2,0)$.

   $\boxed{(a)}$ $(1,2,3)$, (b) $(2,3,4)$, (c) $(3,4,5)$, (d) $(4,5,6)$ (e) None of these.

4. **[2 marks]** Which of the following represents the parametric equations of a line segment through the points $(-1,0,1)$ and $(2,-1,0)$?

   $\boxed{(a)}$ $x=3t-1, \quad y=-t, \quad z=1-t, \quad t\in[0,1]$, (b) $x=2t-1, \quad y=t, \quad z=2-2t, \quad t\in[0,1]$,
   (c) $x=t-1, \quad y=0, \quad z=1+t, \quad t\in[0,1]$, (d) $x=-1-t, \quad y=-t, \quad z=1-t \quad t\in[0,1]$.
   (e) None of these

5. **[2 marks]** Find the volume of the parallelepiped formed by the three vectors $\mathbf{z} = (1, -1, 0)$ and the two base vectors $\mathbf{v} = (0, 1, 2)$, $\mathbf{w} = (-1, 0, 1)$

   (a) 1 (b) 2 (c) 4 $\boxed{(d)}$ 3 (e) None of these.

6. **[2 marks]** An *active rotation* rotates a vector about its initial point.

   $\boxed{(a)}$ TRUE (b) FALSE

PART II: Show all work here and give details.
No additional pages will be accepted

6. **[3 + 5 marks]** a) Find a parametric form of the circle with center $(1, -1)$ and radius $R = 2$ with counterclockwise orientation.

   $\boxed{Answer:}$ $x = 1 + 2\cos t, \quad y = -1 + 2\sin t,\ t \in [0, 2\pi]$.

   b) Find a parametric form for the parabola $4y^2 - 16y + x + 13 = 0$ and specify the parameter interval.

   $\boxed{Answer1:}$ Completing the square in y we get the equivalent form $4(y-2)^2 = 3 - x$. Start with $y = t$, THEN find x. So, $x = 3 - 4(t-2)^2, y = t$, $-\infty < t < \infty$.

   $\boxed{Answer2:}$ Also possible but more difficult to describe is the parametrization where $x = t$. In this case, since $4(y-2)^2 = 3 - x = 3 - t \geq 0$ we must have $t \leq 3$. Solving for y we get

$$y = 2 \pm \frac{\sqrt{3-t}}{2},$$

   where the "upper/lower" part of the parabola (relative to the line $y = 2$) is described using the $+/-$ sign respectively. Observe that the parametrization $x = t,\ y = 2 - \frac{\sqrt{3-t}}{2}$, describes the lower part of the parabolic arc as t varies from $-\infty$ to 3 in the counterclockwise sense. So this is a good parametrization.

   What about the upper arc? As t varies from $-\infty$ to 3 the parametrization $x = t,\ y = 2 + \frac{\sqrt{3-t}}{2}$, describes the upper arc in the **clockwise** sense! One way to fix this is by setting $x = -t$ so that the "old" t-interval $-\infty < t \leq 3$ becomes the "new" t-interval $-3 \leq t < \infty$ (obtained by multiplying $-\infty < t \leq 3$ throughout by -1 and changing the sign of the inequalities). Now we see that the parametrization $x = -t,\ y = 2 + \frac{\sqrt{3+t}}{2}$, $-3 \leq t < \infty$ describes the upper arc in the counterclockwise sense!

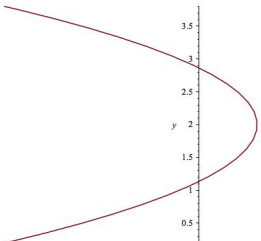
   So, the final (counterclockwise) parametrization of the full parabolic arc is given by

$$x = t, \quad y = 2 - \frac{\sqrt{3-t}}{2}, \quad \infty < t \leq 3,$$

   for the lower arc and

$$x = -t, \quad y = 2 + \frac{\sqrt{3+t}}{2}, \quad -3 \leq t < \infty,$$

   for the upper arc.



Recall that changing t to $-t$ usually *reverses* the orientation of an arc! That's why we used it here.

7. **[5+5 marks]** a) Find the image (x', y') of the vector $(x, y) = (2, -2)$ in standard position when it is rotated counterclockwise by the angle $\theta = \pi/3$.

Answer :

$$x' = 2\cos(\pi/3) + 2\sin(\pi/3), \quad y' = 2\sin(\pi/3) - 2\cos(\pi/3).$$

So,

$$x' = 1 + \sqrt{3}, \quad y' = \sqrt{3} - 1.$$

Note that the length of the vector $(2, -2)$, i.e., $\sqrt{8}$ is unchanged for the new vector, as it must be!

b) Find the coordinates (x', y') of the given point $(x, y) = (4, 0)$ in standard position under the change of origin to $(1, 1)$ followed by a counterclockwise rotation of the new axes by the angle $\theta = 5\pi/6$.

Answer : $(-\frac{3\sqrt{3}+1}{2}, -\frac{3-\sqrt{3}}{2}) = (-3.098, -0.634)$.

Why? Here $(x, y) = (4, 0)$, $(h, k) = (1, 1)$, $\theta = 5\pi/6$. So,

$$x' = 3\cos(5\pi/6) - \sin(5\pi/6), \quad y' = -3\sin(5\pi/6) - \cos(5\pi/6).$$

So,

$$x' = -\frac{3\sqrt{3}}{2} - \frac{1}{2}, \quad y' = -\frac{3}{2} + \frac{\sqrt{3}}{2},$$

and this is the same as the answer above.