Problem 7.(5 points) Find a QR factorization of the matrix $A = \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Set, $V_1 = x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then by G-S process

$$V_2 = x_2 - \frac{x_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}. \text{ Here } V_1 \perp V_2$$

Now, $Q = \left(\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|} \right) = \begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix}$

$$\therefore R = Q^T A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{pmatrix} //$$

Problem 8.(5 points) $A = \begin{pmatrix} 16 & 15 \\ 0 & 1 \end{pmatrix}$ with eigenvalues $\lambda_1 = 16$, $\lambda_2 = 1$ corresponding to eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively. Find a matrix C such that $C^2 = A$.

Here, A is diagonalizable and we can write.

$A = PDP^{-1}$. where, $D = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Let, $B^2 = D \Rightarrow B = \begin{bmatrix} \sqrt{16} & 0 \\ 0 & \sqrt{1} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

Let, $C = PBP^{-1}$ then,

$$C^2 = PB\underbrace{P^{-1}P}_{I}BP^{-1} = PB^2P^{-1} = PDP^{-1} = A.$$

$$\therefore C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} //$$