MAT 1341D Practice Final Winter 2014

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1. **[1 point]** Suppose A is a square matrix with n rows (where $n \geq 1$) and that the rank of A is $n-1$. Which of the following statements is then necessarily true?

   (a) A is not diagonalizable

   (b) The columns of A are linearly independent

   (c) 1 is an eigenvalue of A

   (d) The determinant of A is equal to 1

   (e) A has an eigenvalue $\lambda = 0$.

   (f) The reduced row echelon form of A is equal to the identity matrix I_n

   Solution. If the rank is less than the size of the matrix then it is not invertible. This is turn is equivalent to $det(A) = 0$. Now it follows that $det(A - 0I) = det(A) = 0$, which says that 0 is an eigenvalue of A.

2. **[1 point]** Consider the system $\begin{bmatrix} 1 & 0 & 2 \\ a & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$. For which value(s) of a does the system have infinitely many solutions?

   (a) $a = 3$

   (b) $a \neq -1$

   (c) $a = -1$

   (d) $a = 5$

   (e) $a \neq 3$

   (f) No such value of a exists.

   Solution. Note first that if $a = 0$ then the system has a unique solution (use row-reduction to see that). Assuming $a \neq 0$, we get the following:

$$\begin{bmatrix} 1 & 0 & 2 & | & 3 \\ a & 1 & 1 & | & 4 \\ 0 & 1 & 3 & | & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & | & 3 \\ 0 & 1 & 1-2a & | & 4-3a \\ 0 & 1 & 3 & | & 7 \end{bmatrix}.$$

   The system has rank 2 when the second and third row are scalar multiples of each other; in that case the system has infinitely many solutions. This happens when $1 - 2a = 3$ and $4 - 3a = 7$, which amounts to $a = -1$.

3. [**1 point**] Suppose A is a 1000x999 matrix with rank 998, and that $\vec{b}$ is a column vector. What can we conclude about the system $A\vec{x} = \vec{b}$?

   (a) The system has infinitely many solutions and there will be one parameters.

   (b) The system has infinitely many solutions and there will be two parameters.

   (c) The system has infinitely many solutions and there will be 998 parameters.

   (d) The system has a unique solution.

   (e) The system is inconsistent.

   (f) None of the above.

   Solution. The system has 1000 equations and 999 variables. The REF of A will have one column without a leading one; the corresponding variable in the system will then be free. Thus, if the system is consistent, we would have one parameter and infinitely many solutions. However, we don't know whether the system is consistent.

4. [**1 point**] Suppose A is an nxn matrix whose row echelon form has leading 1s in every column except for the last. Which statement about the set of columns of A is correct?

   (a) It spans $\mathbb{R}^n$ but is not a basis.

   (b) It is linearly independent but is not a basis.

   (c) It is a basis for $\mathbb{R}^n$, but it is not linearly independent.

   (d) It is a basis for $\mathbb{R}^n$, but it doesn't span $\mathbb{R}^n$.

   (e) It spans a proper subspace of $\mathbb{R}^n$ but is not a basis for that subspace.

   (f) It is linearly independent, but it spans a proper subspace of $\mathbb{R}^n$.

   Solution. The first $n - 1$ columns of A are linearly independent and span a proper subspace of $\mathbb{R}^n$ (indeed, form a basis for that subspace). However, adding the last column makes the set of columns linearly dependent, and while it still spans that same subspace, is no longer a basis.

5. [**1 point**] Suppose the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is 4. What is the determinant of

$\begin{bmatrix} 3a-2b & 3d-2e & 3g-2h \\ b-c & e-f & h-i \\ \frac{1}{2}a+b & \frac{1}{2}d+e & \frac{1}{2}g+h \end{bmatrix}$?

(a) -4

(b) 4

(c) -12

(d) 12

(e) -16

(f) 16

Solution. Note that not only have row-reduction steps been done, the matrix has also been transposed. Fortunately, $det(A) = det(A^T)$. To get from A^T to the given matrix, we do the following row operation steps (only demonstrated for the first column, the others are similar):

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \sim \begin{bmatrix} a \\ c \\ b \end{bmatrix} \sim \begin{bmatrix} a \\ b-c \\ b \end{bmatrix} \sim \begin{bmatrix} 3a-2b \\ b-c \\ \frac{4}{3}b \end{bmatrix} \sim \begin{bmatrix} 3a-2b \\ b-c \\ \frac{1}{2}a+b \end{bmatrix}.$$

The first step changes rows, so changes the determinant by a sign. The second step multiplies the second row by -1 and adds the third row; this changes the determinant again by a sign. The third step multiplies row 1 by 3 and then subtracts a multiple of another row. This means we multiply the determinant by a factor 3. We also multiply row 3 by a factor $\frac{4}{3}$, hence the overall result to the determinant is multiplication by $3 \cdot \frac{4}{3} = 4$. Finally we add $\frac{1}{6}$ times row 1 to row 3; this doesn't change the determinant. Conclusion: the determinant of the matrix is $4 \cdot 4 = 16$.

6. [**1 point**] Consider $\mathbb{M}_{2,2}$, the vector space of all 2x2 matrices. Consider also the set

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Is this a basis for $\mathbb{M}_{2,2}$?

(a) Yes

(b) No, because it doesn't contain the identity matrix.

(c) No, because one of the matrices in this set is a linear combination of the others.

(d) No, because there are not enough matrices in this set to span all of $\mathbb{M}_{2,2}$.

(e) No, because it doesn't contain the zero matrix.

Solution. Yes it is, because it consists of four linearly independent vectors in a space of dimension 4. (To check that they are independent, use the fact that $\mathbb{M}_{2,2}$ is really $\mathbb{R}^4$ in disguise and use the usual test for independence there.)

7. [**1 point**] Consider a non-homogeneous linear system $A\vec{x} = \vec{b}$ with 100 equations and 150 variables. Answer the following questions:

   (i) Can the system be inconsistent?

   (ii) Can the system have a unique solution?

   (iii) Can the system have infinitely many solutions?

   (a) Yes, No, No

   (b) Yes, No, Yes

   (c) Yes, Yes, No

   (d) Yes, Yes, Yes

   (e) No, Yes, Yes

   (f) No, Yes, No

   (g) No, No, Yes

   Solution. It can be inconsistent (e.g., one equation may read $0x_1 + \cdots + 0x_{150} = 12$). It can not have a unique solution, because as soon as it is consistent, there will be at least 50 parameters and hence infinitely many solutions.

8. [**1 point**] Consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Is A diagonalizable (over the real numbers)?

   (a) Yes, because A is upper-triangular.

   (b) Yes, because A has two distinct real eigenvalues.

   (c) Yes, because A has an eigenvalue with multiplicity 2.

   (d) No, because A has no real eigenvalues.

   (e) No, because A is invertible.

   (f) No, because A doesn't have two linearly independent eigenvectors.

   Solution. The matrix has only one eigenvalue, $\lambda = 2$, with multiplicity 2. However, $A - 2I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, which has rank 1 and hence only gives 2-1=1 basic eigenvectors. Thus there aren't enough independent eigenvectors to diagonalize.

9. [**1 point**] Consider the vector space $\mathbb{M}_{3,3}$ of 3×3 matrices, as well as the subset

$$W = \{A \in \mathbb{M}_{3,3} | A^T = -A\}.$$

   Which of the following statements is true?

(a) W is a subspace of $\mathbb{M}_{3,3}$ of dimension 0

(b) W is a subspace of $\mathbb{M}_{3,3}$ of dimension 1

(c) W is a subspace of $\mathbb{M}_{3,3}$ of dimension 2

(d) W is a subspace of $\mathbb{M}_{3,3}$ of dimension 3

(e) W is a subspace of $\mathbb{M}_{3,3}$ of dimension 4

(f) W is not a subspace of $\mathbb{M}_{3,3}$.

Solution. The condition says that $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$. Comparing entries gives $a = e = i = 0$, and $d = -b, g = -c, h = -f$. That means that any matrix satisfying the condition has the form $\begin{bmatrix} 0 & -b & -c \\ b & 0 & -f \\ c & f & 0 \end{bmatrix}$. That means that the subspace consisting of these matrices is spanned by

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

As these three matrices are linearly independent, the subspace has dimension 3.

10. [**1 point**] Consider the following polynomials in $\mathbb{P}_3$ (the vector space of polynomials of degree at most 3): $p_1 = x^2 - 2$, $p_2 = x^3 + x^2 + 3x + 7$, $p_3 = x^3 - 2x^2 + 3x + 1$ and $p_4 = x^3 - 5x^2 + 3x + 3$. Let $U = Span\{p_1, p_2, p_3, p_4\}$. What is the dimension of U?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

(f) 5

Solution. Use the fact that $\mathbb{P}_3$ is $\mathbb{R}^4$ in disguise, and use the standard test there:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & -5 \\ 0 & 3 & 3 & 3 \\ -2 & 7 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 9 & -3 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -12 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus there are three independent vectors, hence the dimension is 3.

11. [**1 point**] Which of the following is a basis for the subspace $\{(x, y, z, w) | 3x - 2y + z = 5w\}$ of $\mathbb{R}^4$?

(a) $\{(3, -2, 1, -5)\}$

(b) $\{(1, 1, 0, 0)\}$

(c) $\{(3, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0)\}$

(d) $\{(3, 0, 0, 0), (0, -2, 0, 0), (0, 0, 1, 0), (0, 0, 0, -5)\}$

(e) $\{2, 3, 0, 0), (1, 0, -3, 0), (5, 0, 0, 3)\}$

(f) None of the above.

Solution. The subspace in question is the set of solutions to the homogeneous system $3x - 2y + z - 5w = 0$. This system has 3 parameters and one leading variable. The solutions are thus $\begin{cases} w = s \\ z = t \\ y = r \\ x = \frac{2}{3}r - \frac{1}{3}t + \frac{5}{3}s \end{cases}$, or $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = r \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{5}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$.
Now the three basic solutions form a basis for the subspace.

12. [**1 point**] Consider a linear transformation $T : \mathbb{M}_{2,2} \to \mathbb{M}_{2,2}$ satisfying
$T \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $T \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $T \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. What is $T \begin{bmatrix} 3 & -3 \\ 5 & 4 \end{bmatrix}$?

(a) $\begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -8 & 1 \\ 1 & 2 \end{bmatrix}$

(d) None of the above.

Solution. Use the fact that $\mathbb{M}_{2,2}$ is $\mathbb{R}^4$ in disguise. Then it becomes: $T \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$,
$T \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $T \begin{bmatrix} 2 \\ 1 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$. What is $T \begin{bmatrix} 3 \\ -3 \\ 5 \\ 4 \end{bmatrix}$? However, the vector $(3, -3, 5, 4)$ is not a linear combination of the vectors given, so there is not enough information to answer the question.

13. **[1 point]** For which value(s) of b is $(5, b, 2, -2)$ in the span of

$$\{(1,0,2,0),(1,1,0,3),(1,2,-1,1),(1,1,1,-2)\}?$$

(a) $b = 2$

(b) $b \neq 2$

(c) $b = 6$

(d) $b \neq 6$

(e) For all values of b.

(f) For no value of b.

Solution. To determine whether a given vector is in the span of a set of vectors, solve the corresponding linear system:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 0 & 1 & 2 & 1 & | & b \\ 2 & 0 & -1 & 1 & | & 2 \\ 0 & 3 & 1 & -2 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 0 & 1 & 2 & 1 & | & b \\ 0 & 2 & 3 & 1 & | & 8 \\ 0 & 3 & 1 & -2 & | & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 0 & 1 & 2 & 1 & | & b \\ 0 & 0 & -1 & -1 & | & 8-2b \\ 0 & 0 & -5 & -5 & | & -2-3b \end{bmatrix}$$

We see that the system is consistent if and only if $5(8-2b) = -2-3b$, that is, when $42 = 7b$, i.e., when $b = 6$. Thus for $b = 6$ the given vector is in the span. For any other value of b the system is inconsistent, hence the given vector is not in the span.

14. **[1 point]** Consider the following subsets of the vector space $V = \mathbb{F}[\mathbb{R}, \mathbb{R}]$ of functions from the reals to the reals. Which of these are subspaces? Give a one-line motivation for each.

$$\begin{aligned} U_1 &= \{f \in V | f(x) \neq 0, \text{ for all } x\} \\ U_2 &= \{f \in V | f(12) \geq 0\} \\ U_3 &= \{f \in V | f(2) + f(3) = 0\} \\ U_4 &= \{f \in V | f(x) = -2f(-x) \text{ for all } x\} \end{aligned}$$

Solution. U_1 is not a subspace because it does not contain the zero function $f(x) = 0$.

U_2 is not a subspace because it's not closed under scalar multiplication. For example the constant function $f(x) = 12$ is in U_2, but $-f$ is not.

U_3 and U_4 are subspaces, because they are described using linear equations.

15. **[2 points]** For each of the following matrices, state whether it is diagonalizable. Give a one-line justification for your answers.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Solution. A has eigenvalues $0, 1$, but for $\lambda = 1$, the matrix $A - I$ has rank 2 and therefore only one basic eigenvector. B has eigenvalue 0 but only one basic eigenvector. C is diagonalizable; it's easy to see that $(1, -1, 0), (1, 0, -1), (1, 1, 1)$ are three independent eigenvectors. D has three distinct eigenvalues and is therefore diagonalizable.

16. **[2 points]** Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which first performs a reflection in the plane $z = 0$, and then a rotation around the x-axis, counterclockwise by an angle of $\frac{3\pi}{4}$.

Solution. Test on the standard basis vectors. $T(1,0,0) = (1,0,0), T(0,1,0) = (0, -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}), T(0,0,1) = (0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. Thus the matrix is

$$\frac{1}{2}\sqrt{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

17. **[5 points]** Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 3 \\ 0 & a & -1 \end{bmatrix}$.

(a) Calculate the determinant of A, and use this to determine for which values of a the matrix A is invertible.

(b) Using the Gauss-Jordan algorithm, find the inverse of A.

Solution. Expand along the first row: $det(A) = 1(-2 \cdot -1 - 3a) + 2(1a) = 2 - a$. Thus for $a \neq 2$, the matrix is invertible.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 0 & a & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & -1 & 0 \\ 0 & a & -1 & 0 & 0 & 1 \end{array}\right].$$

Now if $a = 0$ we continue

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

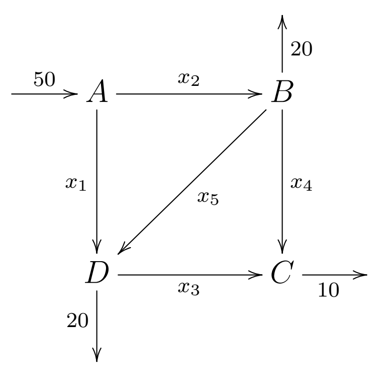
while for $a \neq 0$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{a-2}{2} & -\frac{a}{2} & \frac{a}{2} & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{a}{a-2} & \frac{a}{a-2} & \frac{2}{a-2} \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1+\frac{2a}{a-2} & \frac{-2a}{a-2} & \frac{-4}{a-2} \\ 0 & 1 & 0 & \frac{-1}{a-2} & \frac{1}{a-2} & \frac{1}{a-2} \\ 0 & 0 & 1 & -\frac{a}{a-2} & \frac{a}{a-2} & \frac{2}{a-2} \end{array}\right]$$

Note: this is heavier in calculation than I'd ask on the actual exam, but you should understand the method.

18. **[6 points]** Consider the network of streets and intersections below. The arrows indicated the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down a system of linear equations which describes the traffic flow. Also write down all the constraints on the variables $x_1, \dots, x_5$. (Do not perform any operations on the equations; this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Solution. The constraints are simply that each $x_i \geq 0$ (one-way streets) and that they are integers (no fractions of cars). For the equations, you set, for each intersection, the inflow equal to the outflow.

Intersection	**Inflow**	**Outflow**	**Equation**
A	50	$x_1 + x_2$	$x_1 + x_2 = 50$
B	x_2	$x_4 + x_5 + 20$	$x_2 - x_4 - x_5 = 20$
C	$x_3 + x_4$	10	$x_3 + x_4 = 10$
D	$x_1 + x_5$	$x_3 + 20$	$x_1 - x_3 + x_4 = 20$

(b) The row echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 50 \\ 0 & 1 & 1 & 0 & -1 & 30 \\ 0 & 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution of this system. (Ignore the constraints at this point.)

Solution.

$$\begin{cases} x_5 &= t \\ x_4 &= s \\ x_3 &= 10 - s \\ x_2 &= -x_3 + x_5 + 30 \\ &= 20 + s + t \\ x_1 &= 50 - x_2 \\ &= 30 - s - t \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

(c) If the road from A to D were closed due to roadwork, find the maximum and minimum flow along the road from B to D, **using your results from part (b)**.

Solution. If $x_1 = 0$, then $30 - s - t = 0$, i.e. $s = 30 - t$. This gives a solution of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = (30 - t) \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 50 \\ -20 \\ 30 \\ 0 \end{bmatrix}$$

The third entry says that $t \geq 20$ (because we must have positive flow) and the fourth says that $t \leq 30$. Thus $20 \leq t \leq 30$. Since $x_5 = t$, this means that the minimum flow along BD is 20 and the maximum flow is 30.

19. [**6 points**] Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A, and find the eigenvalues of A.

(b) For each eigenvalue λ of A, find the corresponding eigenvectors of A, and give a basis for $E_\lambda = \{\vec{x} \in \mathbb{R}^3 | A\vec{x} = \lambda\vec{x}\}$.

(c) If possible, give a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. If this is not possible, explain why.

Solution.

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{bmatrix} = (1-\lambda)[(1-\lambda)(-1-\lambda)+1] = (1-\lambda)\lambda^2.$$

Thus the eigenvalues are $\lambda = 1$ and $\lambda = 0$.

For $\lambda = 1$, we find

$$A - I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

with eigenvectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for E_1.

For $\lambda = 0$, we find

$$A - I = A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We get $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for E_0.

We only find two independent eigenvectors, which is not enough to diagonalize. (One needs 3 independent eigenvectors to diagonalize a 3x3 matrix.)

20. [**6 points**] Consider the subspace

$$U = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}_4 | u_1 - 2u_2 + u_3 - 2u_4 = 0\}.$$

(a) Find a basis for U and give the dimension of U.

(b) Use the **Gram-Schmidt Algorithm** to find an orthogonal basis for U.

(c) Find the best approximation by a vector in U to the vector $(1, 0, 0, -1)$.

Solution.

(a) The general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The three basic solutions constitute a basis for U.

(b) Set $v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Next,

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \tfrac{-2}{5} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}. \text{ It's easier to replace this by } v_2 = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \end{bmatrix}.$$

$$v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \tfrac{4}{5} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \tfrac{-2}{30} \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}. \text{ Replace by } v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -2 \end{bmatrix}.$$

Now $\{v_1, v_2, v_3\}$ is an orthogonal basis for U.

(c)

$$\begin{aligned} Proj_U(1, 0, 0, -1) &= Proj_{v_1}(1, 0, 0, -1) + Proj_{v_2}(1, 0, 0, -1) + Proj_{v_3}(1, 0, 0, -1) \\ &= \frac{1}{5}v_1 + \frac{-3}{30}v_2 + \frac{3}{15}v_3 \\ &= \begin{bmatrix} \frac{7}{10} \\ \frac{6}{10} \\ \frac{-3}{10} \\ \frac{-6}{10} \end{bmatrix} \end{aligned}$$