6. Let $M_{2\times2}(\mathbb{R})$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{pmatrix} a & b \\ a-b & a+c \end{pmatrix} \in M_{2\times2}(\mathbb{R}) \mid a,b,c \in \mathbb{R} \right\}.$$

(6a) Show that U is a linear subspace of $M_{2\times2}(\mathbb{R})$. (1)

$U = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ✓

Can be written as a spanning set and therefore is a subspace

1/1

(6b) Find a basis for U, and, hence, find the dimension of U. (2)

$\text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ → none of these matrices are liner combinations of the other ones.

Due to the locations of the zeros and ones in the matrices, there exists no non-zero scalars s.t. $am_1 + bm_2 + cm_3 = 0$, so this set is L.I. and so forms the basis

$$\text{Basis} \therefore \left\{ \underset{1}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}, \underset{2}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \underset{3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$ ✓

2/2

The basis contains 3 elements so

Dim U = 3

5. Consider the vector space $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomial functions of degree at most 2, and define the subset

$$W = \{p(x) \in P_2 \mid p(0) = 0\}.$$

(5a) Show that W is a linear subspace of P_2. (1)

1/1

1 Contains zero as $f(0) = 0 \in W$ ✓

2. Closed under addition:
Suppose $g(x) \in W$ and $f(x) \in W$ → adding 2 polynomials results in another polynomial

$g(0) + f(0) = 0$

$0 + 0 = 0$ /

3 Closed under multiplication → multiplying a polynomial by a scalar results in a polynomial

$f(x) \in W$

$kf(0) = 0$ / → any scalar $\cdot\, 0 = 0$

$(kf)(0) = 0$

(5b) Find a basis for W. (2)

(a correct answer without explanations is (1) only)

Set of vectors with a root at zero $(x-0)(ax+b)$

$(x)(ax+b)$

2/2

$ax^2 + bx$

$= a(x^2) + b(x)$

$W = \operatorname{Span}\{x^2, x\}$ → there is no scalar s.t $kx = x^2$ or $kx^2 = x$

Claim that $\{x^2, x\}$ is LD, this means there exists non-zero scalars $a, b \in \mathbb{R}$ s.t

$ax^2 + bx = 0$

$ax = -b$ → there is no scalar that will give $-b$ and x

This implies that $a = b = 0$ so set is L.I and forms the basis

$\boxed{\text{Basis}: \{x^2, x\}}$

3. It is known that a subspace Y of $\mathbb{R}^{99}$ can be spanned by 75 vectors and that Y has a linearly independent set with 63 vectors. Then it is always true that: (1)

`cross (X) the correct answer:`

- A $63 < \dim Y < 99$
- B $63 \leq \dim Y < 75$
- X $63 \leq \dim Y \leq 99$
- D $75 \leq \dim Y \leq 99$
- E $\dim Y < 63$
- F $\dim Y > 63$

Spanning set of 75 vectors could be LI so Dim Y could be 75 so B is not guarantee.

minimum dimension is 63, but it could be 63 so E, F, A, D are wrong

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4. Suppose $\{u, v\}$ is a linearly independent set in a vector space V, and that $w \in V$ is chosen so that $\{u, v, w\}$ is linearly independent. Which of the following statements is ALWAYS true? (1)

`cross (X) the correct answer:`

- X $\{u, w\}$ is linearly independent
- B $\{v, w\}$ is linearly dependent
- C $u \in Span\{v, w\}$
- D $v \in Span\{u\}$
- E $w \notin Span\{v, w\}$
- F $V = Span\{u, v, w\}$

$w \in Span\{u, v\}$

$(1,0), (0,1)$ $(1,k)$

$(0,1) \in Span\{(1,1),(1,0)\}$

(7c) If v_1, v_2, v_3 are non-zero vectors in a vector space V, and $U = Span\{v_1, v_2, v_3\}$, then $\dim U = 3$.

Answer (T/F): $\boxed{F}$ ✓ ✓ (1)

Justification: (1)

Suppose $U = Span\{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}, \overset{v_3}{(1,1)}\}$

$(1,1) = (1,0) + (0,1)$

$v_3 \in Span\{v_1, v_2\}$

So Basis: $\{(1,0)\ (0,1)\}$

2 elements mens that $\dim U = 2$

2/2

(7d) Let $U = Span\{x\sin^2(x), x\cos^2(x), x\}$ be a subspace of $F(\mathbb{R})$. Then $\dim U = 3$.

Answer (T/F): $\boxed{F}$ (1)

Justification: (1)

2/2

$0 = x(a\sin^2 x + b\cos^2 x + c)$ ✓

Set $b=1$ and $a=1$ $\quad x(\sin^2 x + \cos^2 x + c)$

$x(1 + c)$

$c = -1$

$x(1-1)$

$x(0)$

$= 0$

So there exist non-zero scalars $a = b = 1$, $c = -1$ such that $ax\sin^2 x + bx\cos^2 x + cx = 0$

So set is L.D and $\dim U < 3$

7. State whether each of the following statements is always true, or is possibly false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example where it fails.
- If you say the statement is always true, you must give a clear explanation.

(7a) The subset $X = \{f \in F(\mathbb{R}) \mid f(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$ is a subspace of the vector space $F(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$.

Answer (T/F): $\boxed{F}$ ✓ (1)

Justification: (1)

Not closed under scalar multiplication

$f(x) = 1 \in X$ ✓

$-f(x) = -1 \to$ does not belong to X $\frac{2}{2}$

✓

(7b) The subset $U = \left\{ \begin{pmatrix} a & b \\ 1 & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b, c \in \mathbb{R} \right\}$ is a subspace of the vector space of 2 by 2 matrices $M_{2\times 2}(\mathbb{R})$.

Answer (T/F): $\boxed{F}$ ✓ (1)

Justification: $\frac{2}{2}$ (1)

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin U$

1. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid xy + zw = 0\}$. Then, (1)

$(1,0,0,1)$
$(0,1,0,1)$

`cross (X) the correct answer:`

- [A] U is closed under addition and U is closed under multiplication by scalars
- [B] $(0, 0, 0, 0) \in U$ and U is closed under addition
- [C] U is not closed under addition and U is not closed under multiplication by scalars
- [D] U is closed under addition but U is not closed under multiplication by scalars
- [X] $(0, 0, 0, 0) \in U$ and U is closed under multiplication by scalars
- [F] None of the other statements is true

$2(1,0,0,1) \in U$
$k(0,1,0,1) \in U$

---

2. Which of the following are subspaces of $\mathbb{R}^3$? (1)

$U = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 3y + z = 0\}$ ✓ $0 \in U$ a plane through $(0,0)$

$V = \{(x, y, z) \in \mathbb{R}^3 \mid (x - z)(x - y) = 0\}$ $x^2 - xy - zx + zy = 0$

$W = \{(x + y, 2y, x - y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$ ✓ $\operatorname{Span}\{(1,0,1), (1,2,-1)\}$

$X = \{(x, xy, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

$x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}1\\2\\-1\end{pmatrix} = x+y,\ 2y,\ x-y$

`cross (X) the correct answer:`

- [A] Only U and V
- [X] Only U and W
- [C] Only U and X
- [D] Only V and W
- [E] Only V and X
- [F] Only W and X

(6c) Give a basis for U different from the one you gave in (6b). (2)

Basis $\left\{ \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

These 3 matrices are L.I and thus form a different basis for U. It contains 3 elements so Dim = 3

✓ 2/2