Test 1, SOLUTION MATH 1104E, Fall, 2017

Last name: First name: Student no.:

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1. [2 marks]: Circle ALL matrix (matrices) which is (are) in REF (Row Echelon Form):

$$A = \begin{bmatrix} 2 & 0 & 7 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 & 7 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 6 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Answer: A, B, C

2. [2 marks]: Circle ALL matrix (matrices) which is (are) in RREF (reduced row-echelon form)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Answer: All A, B, C

3. [4 marks]: Let A, B and C be 3×3 invertible matrices. Assume that $\det A = 2$ and $\det B = 3$ and $\det C = 9$. What is $\det\left((2A)^T B^2 C^{-1}\right)$?

(a) 6 **(b)** 16 **(c)** 4 **(d)** 72 **(e)** 36 **(e)** None

Answer: (b)

4. [4 marks]: Convert the matrix $\begin{bmatrix} 3 & 3 & -3 & 12 \\ 2 & 3 & -4 & 11 \\ 1 & 2 & -3 & 7 \end{bmatrix}$ to the RREF (reduces row echelon form).

Solution: $R_1' = \frac{1}{3}R_1 \quad \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & -4 & 11 \\ 1 & 2 & -3 & 7 \end{bmatrix} \xrightarrow[R_2'=R_2+(-2)R_1]{R_3'=R_3+(-1)R_1} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} R_3' = R_3 + (-1)R_2$

$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1' = R_1 + (-1)R_2 \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

different approaches should result the same RREF

5. [8 marks]: For what values of h and k does the system $\begin{matrix} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k \end{matrix}$

have: a) no solution, b) infinitely many solutions, c) a unique solution?

Solution: The augmented matrix is $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right]$, replace R_2 by $R_2 + (-3)\,R_1$ the result is an *(REF)* matrix $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$, 2 marks that is,

a) There is no solution, if $h - 9 = 0$ and $k - 6 \neq 0$, that is, if $h = 9$ and $k \neq 6$. 2 marks
b) Infinitely many solutions, if $h - 9 = 0$ and $k - 6 = 0$, that is, if $h = 9$ and $k = 6$. 2 marks
c) Unique solution, if $h \neq 9$ and k can take any real number. 2 marks

6. [8 marks]: Consider the linear system
$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 18 \\ x_1 + x_2 + x_3 &= 7 \\ -x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

a: Write its augmented matrix, then reduce it to a REF form
b: Find the general solution of this system.

Solution: (a): $\begin{bmatrix} 2 & 3 & 1 & 18 \\ 1 & 1 & 1 & 7 \\ -1 & 1 & -3 & 1 \end{bmatrix}$ 1 mark

$$R_1 \longleftrightarrow R_2 \begin{bmatrix} 1 & 1 & 1 & 7 \\ 2 & 3 & 1 & 18 \\ -1 & 1 & -3 & 1 \end{bmatrix} \left(\tfrac{R_2'=R_2-2R_1}{R_3'=R_3+R_1}\right) \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 2 & -2 & 8 \end{bmatrix} (R_3' = R_3-2R_2) \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 marks
the REF form is not unique, they may end up with different forms

(b): Taking $x_3 = t$, from the second row of REF matrix we have $x_2 = 4 + t$. Also, from the first row of REF matrix we have $x_1 = 7 - x_2 - x_3 = 3 - 2t$. So, the solution of the system is $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-2t \\ 4+t \\ t \end{bmatrix} = \textbf{OR} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. 4 marks

7. [6 marks]: Find the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \\ 3 & 3 & 1 & 4 \end{bmatrix}$

Solution: Subtract column 1 from columns 2, 3, and 4, we have: $\det A = \begin{vmatrix} 1 & 1 & 2 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 \\ 3 & 0 & -2 & 1 \end{vmatrix}$

expand along the second row $= -\begin{vmatrix} 1 & 2 & 5 \\ 1 & 3 & 3 \\ 0 & -2 & 1 \end{vmatrix} = (R_2' = R_2 - R_1) = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 3.$

There are different ways to reach to 3.

8. [6 marks]: Use Cramer's Rule to find x_2 (only) in the following system
$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 - x_2 + x_3 &= 3 \\ 5x_1 + 2x_2 + x_3 &= 12 \end{aligned}$$

Solution: $x_2 = \dfrac{\begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 5 & 12 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 5 & 2 & 1 \end{vmatrix}}$ 2 marks

$= \frac{18}{9} = 2.$ 4 marks
There are different ways to find the above determinants.