Test 2SOLUTIONS MATH 1104 E, Fall, 2017

Last name: First name: Student no.:

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1. [3 marks]: Let A be a 4×7 matrix such that dim(Col$A = 4$). What is dim(Nul A), (dimension of the solution space for $A\mathbf{x} = \mathbf{0}$)?

(a) 1 **(b)** 2 **(c)** 3 **(d)** 4 **(e)** 5 **(f)** None

Answer: **(c)**

2. [3 marks]: Let $T : R^2 \longrightarrow R^2$ be a linear transformation such that

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What is $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$?

(a) $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **(e)** $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ **(f)** None

Answer: **(b)**

3. [5 marks]: Find a value of k such that the given vectors are linearly **dependent**

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ k \\ 1 \end{bmatrix}$$

Solution:

$$\det[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \det \begin{bmatrix} -1 & 2 & 3 \\ 1 & -3 & k \\ 3 & 4 & 1 \end{bmatrix} = 40 + 10k = 0 \Longrightarrow k = -4$$

An alternative way: reduce the matrix $[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ to REF, then find k such that at least one column of the reduced matrix is not leading column. In this case an REF form of is $\begin{bmatrix} -1 & 2 & 3 \\ 1 & -3 & k \\ 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & k+3 \\ 0 & 0 & 10k+40 \end{bmatrix}$. Put $10k + 40 = 0 \Longrightarrow k = -4$

4. [5 marks]: Determine wether the following vectors are linearly independent or dependent

$$\mathbf{x}_1 = \begin{bmatrix} -9 \\ 2 \\ 6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

$$\det[\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3] = \det \begin{bmatrix} -9 & 7 & 8 \\ 2 & 4 & 0 \\ 6 & 0 & 0 \end{bmatrix} = 9 \times 4 \times 6 \neq 0 \Longrightarrow \text{the vectors are linearly independent.}$$

Alternatively: Let $A = [\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3]$, then the only solution to $A\mathbf{c} = \mathbf{0}$, is zero vector. Then the vecotrs are linearly independent.

5. [3+3+4+2+2 marks]: The matrix A and the matrix R, an echelon form of it, are given:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a: Find a basis for column space A.
b: Find a basis for the row space of A.
c: Find a basis for the null space of A, the solution space of $A\mathbf{x} = \mathbf{0}$.
d: What is the rank A?
e: Verify the rank Theorem

Solution:

a: Columns 1, 2 and 4 of the matrix are R leading columns so the corresponding columns of A form a basis for the col A, that is a basis for Col A is: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

b: A basis for Row A is the set of the first three row vectors of R, that is:

$$(1, 0, 1, 0, 1)\ ,\ (0, 1, -2, 0, 3)\ ,\ (0, 0, 0, 1, -5).$$

c: Find solution for $A\mathbf{x} = \mathbf{0} \iff R\mathbf{x} = \mathbf{0}$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = r$ and $x_5 = s$ are free: $x_4 = 5s,\ x_2 = 2r - 3s,\ x_1 = -r - s \implies$

$\begin{bmatrix} -r-s \\ 2r-3s \\ r \\ 5s \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$, A basis for the null A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

d: rank A=3

e: rank A + nullity A= 3 +2 =5= number of columns of A.

6. [2+2+3+3 marks]: Let $T : R^3 \longrightarrow R^3$ is defined by : $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \\ x+2y+z \end{bmatrix}$.

a: Find $T(1, 2, 3)$.
b: Find the standard matrix of T.
c: Is T one-to-one? Why?
d: Is T onto? Why?

Solution:

a: (3 ,5, 8)

b: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

c: An REF form of A is $R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

dim (ker T)= number of free variables in $R\mathbf{x} = \mathbf{0}$, that is $1 \neq 0$, So T is NOT one-to-one.

d: dim(im T)=number of leading entries in R which is $2 \neq$ dim (R^3=3). So T is NOT onto.