Lecture 3.1: Deflections - Introduction

Lecture outline:

1. Deflections
2. Importance of deflections
3. Factors that affect deflections
4. Basic Assumptions
5. Methods to calculate deflections
6. Elastic beam theory
7. Drawing the deflected shape of structures

1. Introduction

In addition to checking structural members or systems for strength (reactions, shear, moment, normal force), deflections must also be checked. Deflections can limit the load capacity of a member or structural system.

Deflections can be affected by many factors including **load**, **temperature**, **settlements** …

$$\Delta_{total} = \Delta_{load} + \Delta_{temp} + \Delta_{settlement}$$

For Δ_{load} , the deflection of the structure will be caused by its internal loads: N, V, M.

- In the analysis of beams the greatest deflections are caused by ***bending*** (M)
    - ... usually we assume beams are axially rigid
- In the analysis of trusses the greatest deformations are caused by ***axial forces***(N)

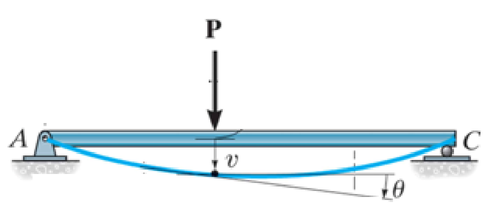
Example for beam:

- Transverse loads
- → Cause internal loads (Internal Moment)
- → produces **transverse displacement** of points along the axis of the beam

$v(x)$ = deflection = the **transverse displacement** along the beam axis

- δ_i or Δ_i or v_i = deflection at a specific point "i"
- $\delta_{\max}$ or $\Delta_{\max}$ or $v_{\max}$ = the maximum deflection

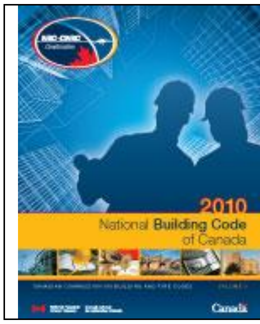
$\theta(x)$ = rotation angle (slope) = the **slope** of the curve that describes the deflection



2. Importance of deflections

Determining deflections in structures is important in the design procedure:

- for **serviceability** requirements ... deflection limits
- structure must not deflect severely in order to **appear to be “safe”** to occupants.



National Building Code of Canada:

- Beam that supports a floor : $\delta_{\max} \leq \frac{L}{360}$
- where L = the span of the beam

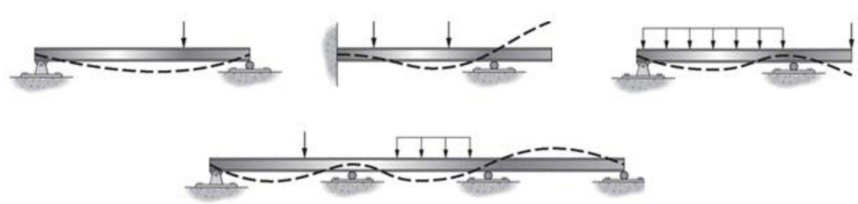
- for preventing damage to **non-structural components** (e.g. cracking of attached brittle materials such as plaster, glass …).

- Limiting **lateral displacements** (non-linear displacements) is important to prevent collapse of structures during earthquakes

Furthermore, as we will see later on in this course, the ability to determine deflections is important in order to be able to analyze statically **indeterminate** structures .

3. Factors that affect deflections

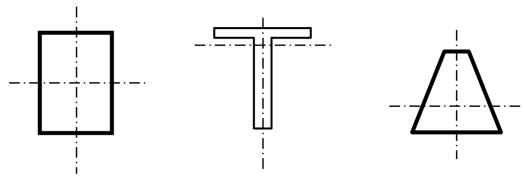
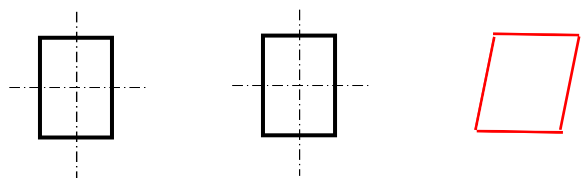
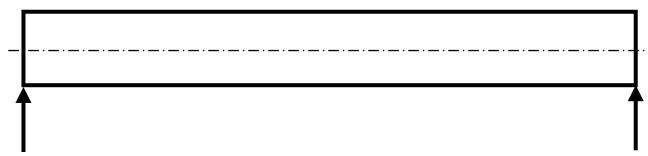
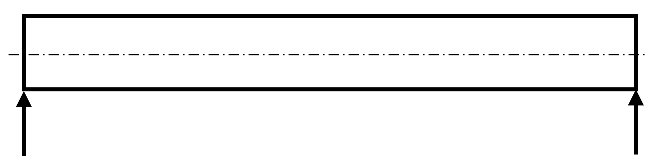
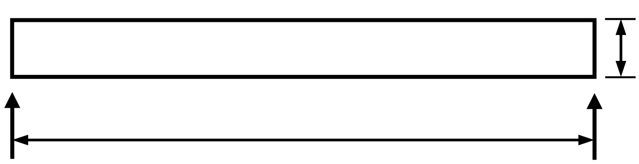
1. Loading (larger load = larger deflection)
2. Nature of supports / Type of beam

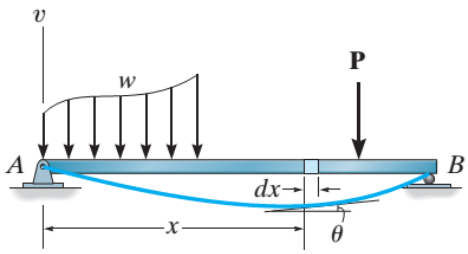


3. Modulus of Elasticity (***E***) and Moment of Intertia (***I***)
   - If ***E*** ⇑ ... structure is more **rigid ... less deformations**
   - If ***I*** ⇑ ... structure is more **rigid ... less deformations**
   - often the term "**Flexural Rigidity**" is used to describe " ***EI*** "
4. Other factors: temperature gradients/changes, settlements

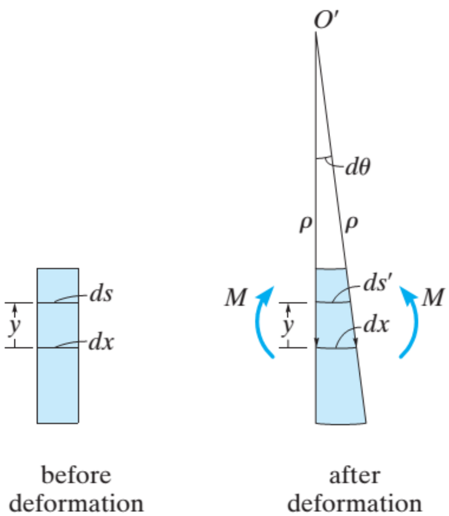
4. Elastic Beam Theory

Assumptions

1)	The member is initially • straight • symmetric cross section with respect to the plane of applied load (no warping)	
2)	Plane sections remain plane through the loading and the deflection process	Before Bending After Bending **Not This** 
3)	The elastic curve does not change length after bending and is defined by the neutral axis	
4)	The displacement of the beam is very small when compared to the member length	
5)	Shear deformations are small when compared to bending deformations. • This is true when the section length to depth ratio (L/d) is greater than 10. Therefore shear deformations can be neglected.	



(a)



(b)

When the internal moment, *M*, deforms an element of the beam

- Each cross section remains plane
- the angle between points on different elevations of the cross section becomes $d\theta$
- The radius of curvature is the distance, ρ, measured from the center of curvature, O', to the neutral axis of the section

Using the above, we can say that the strain at any point (*y*) on the cross section is:

$$\epsilon = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta}$$

Rearranging the above equation we get:

$$\frac{1}{\rho} = -\frac{\epsilon}{y}$$

The above relates the:

- radius of curvature to the strain at any height of the section

As we said that the material is homogeneous and linear elastic, we can apply

- Hooke's law $\epsilon = \sigma/E.$

We can also relate stress in an elastic section to moment and section geometry through

- $\sigma = -My/I$

Combining these, we can now say that:

$$\phi = \frac{1}{\rho} = \frac{M}{EI}$$

Where:
ϕ= Curvature
ρ= Radius of curvature
M= Internal moment on section
E= Modulus of elasticity for material in section
I= Moment of inertia for the section

If we define displacement (*v*) as positive when going up, (see figure (a) on the previous page) and express the curvature in terms of *v* and *x*, we can use calculus to derive a non-linear second order differential equation for curvature as a function of displacement.

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

Since we already defined curvature in terms of moment and rigidity, we now know:

$$\frac{M}{EI} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

This equation is a *general* form of the relationship between deflection and moment. However, it is difficult to solve. By recognizing that:

- dv/dx *is the slope*
- Slope is very, very small for elastic beams
- 1+(dv/dx)^2=1 using small deflection theory

We can reduce our general form of the moment to displaced shape equation to:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

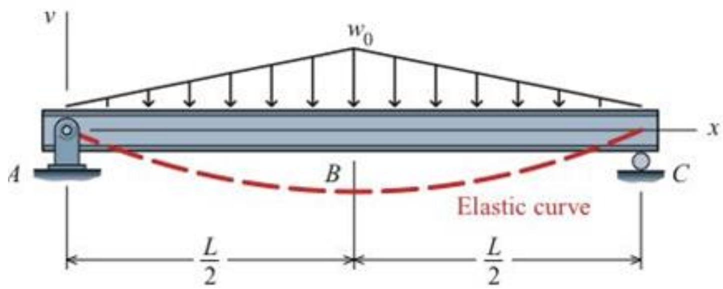
- Recall **double-integration method**:
  - integrate once ... get slope: $\theta \approx \frac{dv}{dx} = \int \frac{M(x)}{EI} dx$
  - integrate twice ... get deflection : $v(x) = \int \int \frac{M(x)}{EI} dx$

Load	Shear	Moment	Rotation	Displacement
$w(x) = \frac{d^2M(x)}{dx^2}$	$V(x) = \frac{dM(x)}{dx}$	$M(x)$	$\theta = \int \frac{M(x)}{EI} dx$	$y = \iint \frac{M(x)}{EI} dx$

5. Sketching the deflected shape (elastic curve) of a structure:

Sketching deflection in beams and frames:
Before computing deflections, there is benefit in qualitatively sketching the expected deformed shape of a structure under load (this is called drawing the **elastic curve**).

Useful before starting an actual analysis ... can be used to "check" computed results.

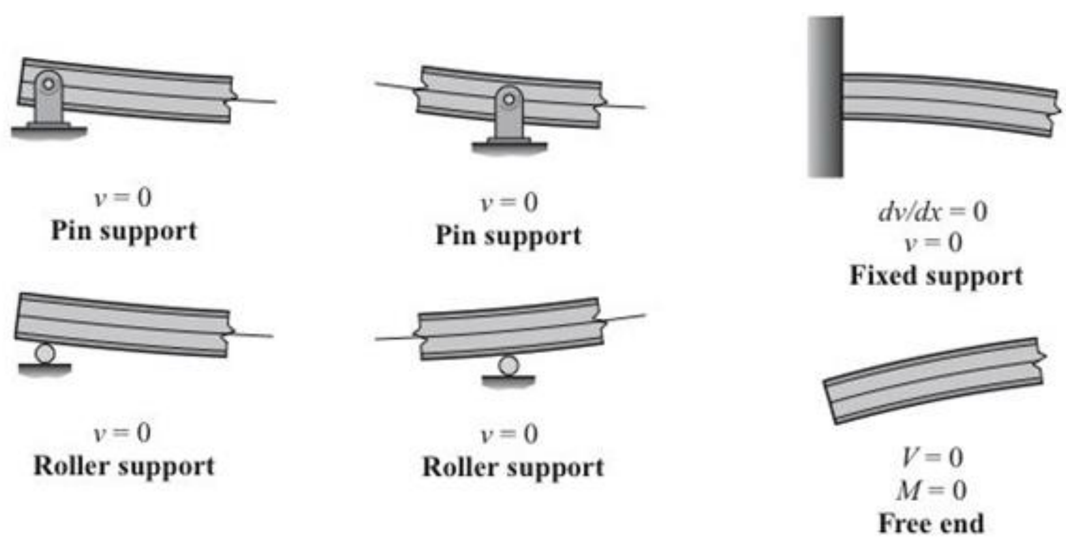


One can use some basic principles to sketch the deformed shape (elastic curve) of a structure.

(1) Principles related to joints/supports in beams

There are some principles related to the way different types of supports will rotate and deflect under load:

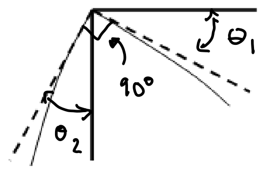
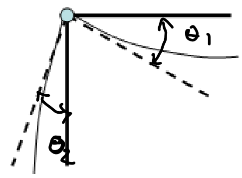
For supports:
- supports that resist a force such as a "**pin**" support restrict **displacement**
- supports that resist a moment such as a "**fixed**" support restrict **rotation** (in addition to displacement)



(2) Principles related to joints/supports in frames

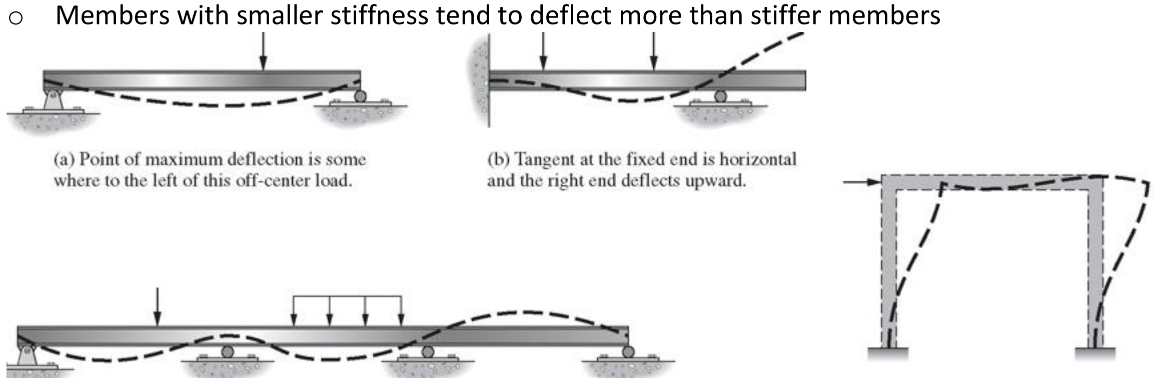
For frame members:

- joints that are "**rigid**" .. cause joint to rotate the members by the **same** amount θ.
- joints that are "**pin**" ... cause members to rotate by a **different** amount θ at the joint since the pin cannot support a moment.

	Rigid joint connection $\theta_1 = \theta_2$
	Pin joint connection $\theta_1 \neq \theta_2$

(3) Principles related to members

- A member deforms in the direction of the load applied to it
- Deflections of loaded members are sketched first.
  - Then one tries to understand how the joints will deflect/rotate.
    - Finally the deflections of unloaded members are sketched
- Unless there is a hinge between a member and a joint, the end of the member and joint displace in the same manner
- Members with smaller stiffness tend to deflect more than stiffer members



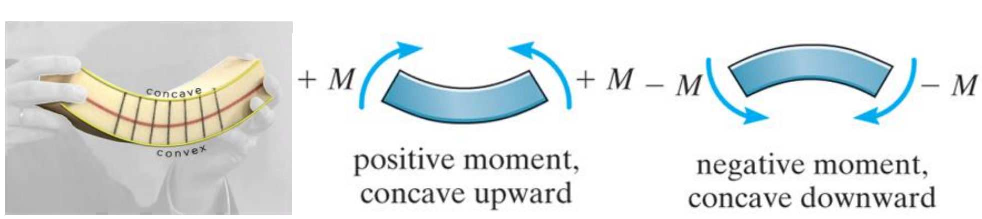
(a) Point of maximum deflection is somewhere to the left of this off-center load.

(b) Tangent at the fixed end is horizontal and the right end deflects upward.

(d) Note the upward deflection in the third span.

(e) The frame sways to the right.

(4) There is a direct link between curvature and moment (see next page)

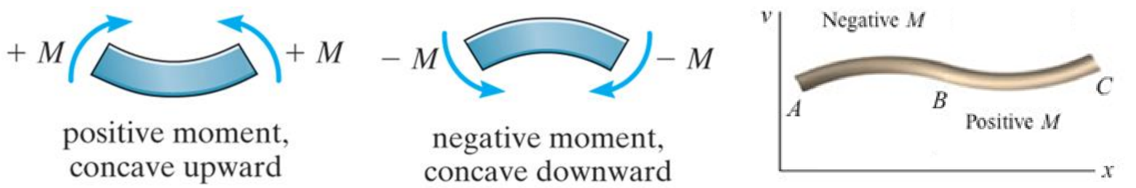


The link between curvature and moment ....

Recall: $\phi = \dfrac{d^2 v}{dx^2} = \dfrac{M}{EI}$

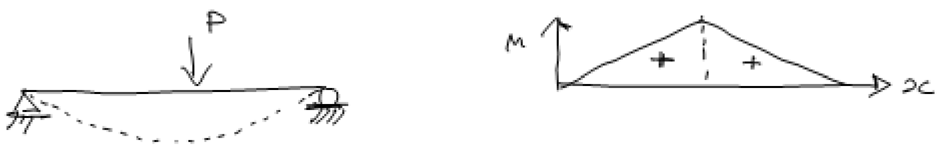
If you have difficulty drawing the deflected shape of the structure it is useful to know that curvature can be related to the moment diagram. On the flip side, if you can determine the deflected shape this can help you determine the qualitative shape of the moment diagram !

Recalling our positive sign convention....
and recalling that moments are drawn on the compression side of members ...



If M=**positive (+)**: then curvature will be positive; deflected shape will be "concave upward" (☺)

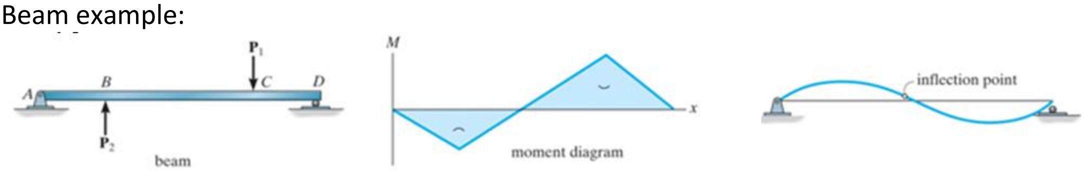
- similarly if I know curvature is positive I know moment will be positive



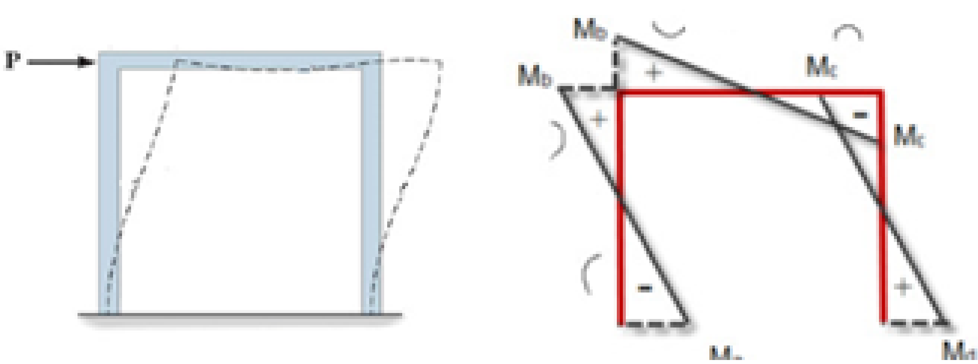
If M=**negative (-)**: then curvature will be negative; deflected shape will be "concave down" (☹)

- similarly if I know curvature is negative I know moment will be negative

Beam example:

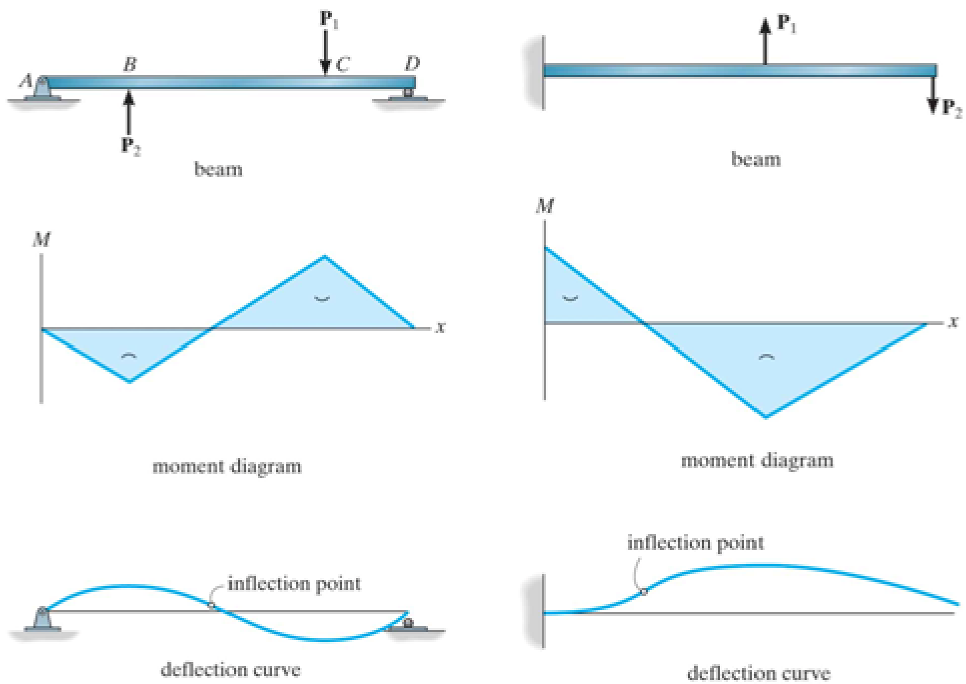


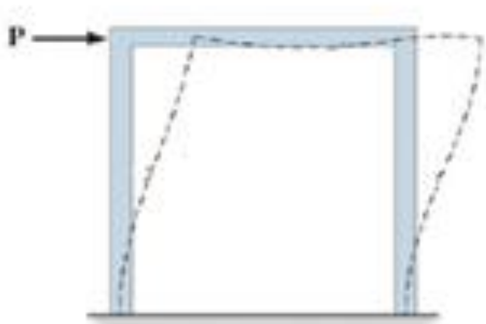
Frame example:

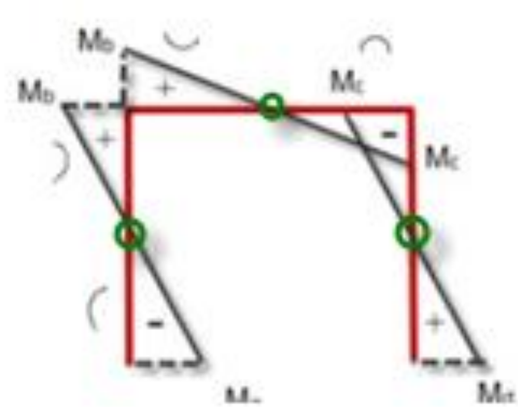


Another important observation is that a location with **zero moment** will be an "**inflection point**" :

- similarly if I can guess the location of inflection points, I can use this to help draw the moment diagram since I know ,M=0 at these locations







Lecture 3.2: Deflections - Conjugate beam & Moment-area

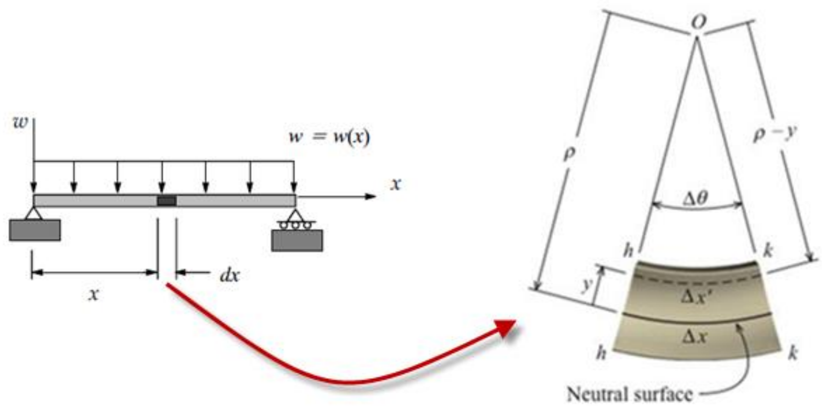
Lecture outline:

1. Assumptions and basic methods to calculate deflections
2. Conjugate beam method
3. Moment-area method

1. Assumptions and basic methods to calculate deflections

Recall the relationship between curvature and moment:

$$\phi = \frac{1}{\rho} = \frac{d^2 v}{dx^2} = \frac{M}{EI}$$



Methods to calculate deflections:

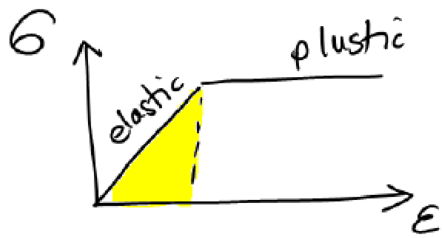
There are several methods that can be used to calculate elastic deflections:

1. Double Integration .... covered in CVG2140
2. Singularity functions .... covered in CVG2140
3. Moment area-method .... covered in **CVG3140**
4. Conjugate-beam method .... covered in **CVG3140**
5. Virtual-Work method .... covered in **CVG3140**

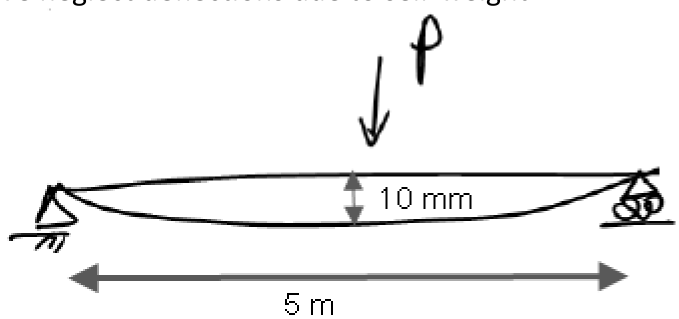
Basic Assumptions:

Assumptions for elastic deflections in this course:

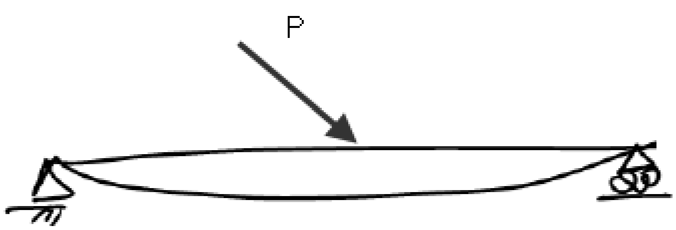
- Hooke's law is valid (**linear-elastic behaviour**): we are only considering elastic deflections



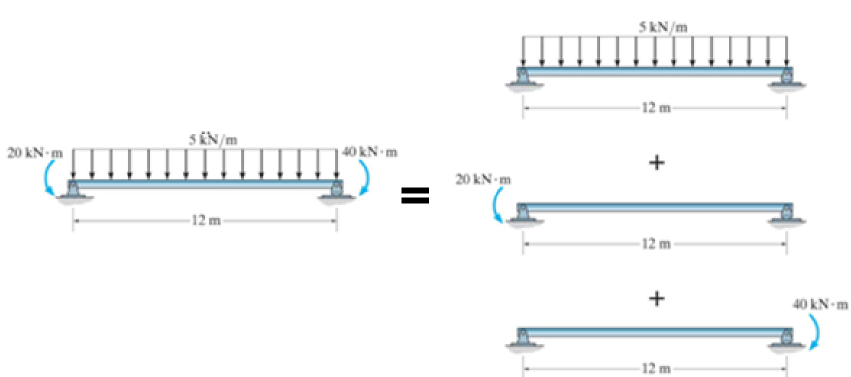
- Deflections are **small** relative to the geometry of the structure
- We neglect deflections due to self-weight



- Axial and shear deformations are small compared to flexural deformations



- The **principle of superposition** is applicable



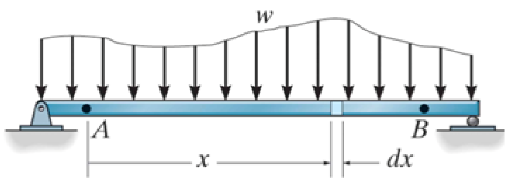
2. Moment-area method

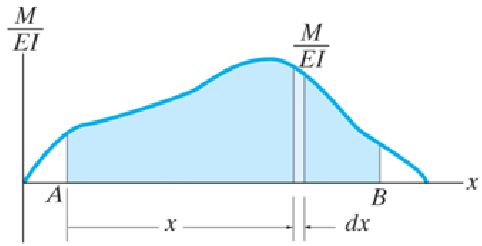
This method was derived in the late 1800s by Otto Mohr and Charles E. Greene. It is a "classical" semi-graphical technique for finding slope and deflection.

The theorems of the method are based on the equation: $\frac{d^2v}{dx^2} = \frac{M}{EI}$

Let us draw the M diagram and divide by the flexural rigidity, *EI* → "M/EI" diagram

- To develop the theorems let us recall our famous equation: $d\theta = \frac{M}{EI}dx$



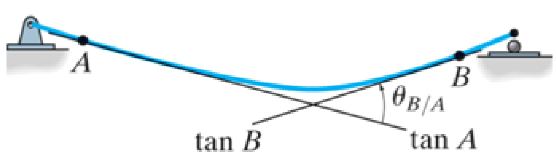


1st-moment area theorem

Integrate $d\theta = \frac{M}{EI}dx$ from point A to point B → $\theta_{B/A} = \int_A^B \frac{M}{EI}dx$

(1): The change in slope** between any two points on the elastic curve equals the **area under the M/EI diagram** between **these two points

- $\theta_{B/A}$ = angle of the tan B measured with respect to the tan A



Notes:

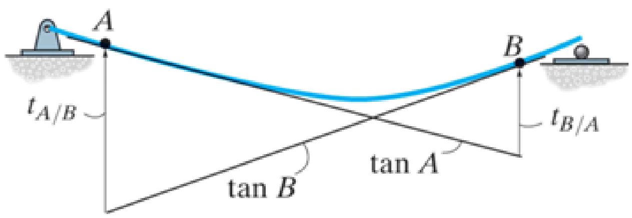
- $\theta_{B/A}$ is measured in radians !
- If area $\frac{M}{EI}$ = positive $(+)$... angle is counter-clockwise from A.
- If area $\frac{M}{EI}$ is negative $(-)$... angle is measured clockwise from A.

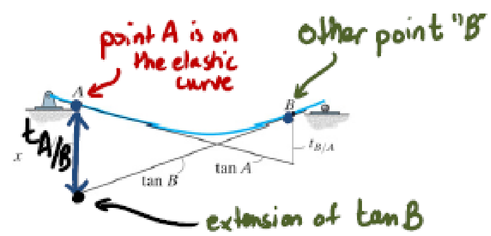
2nd-moment area theorem

The second moment area theorem gives a method of calculating "deflection" if used correctly.
It is based on the relative deviation of the tangents on the elastic curve:

*(2) The **vertical deviation** of **point A on the elastic curve** with respect to a **tan B** equals the "**moment**" of the **area M/EI** between A and B. This moment is **computed about point A**.*

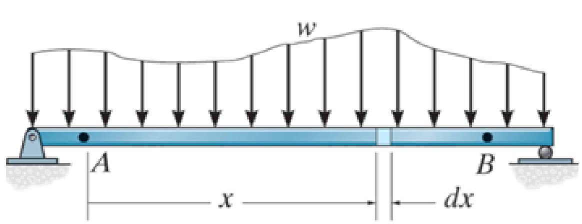
- $t_{A/B}$ = vertical deviation of tan A with respect to tan @ point B
- "moment-arm" is taken from **point A** to the centroid of the area M/EI

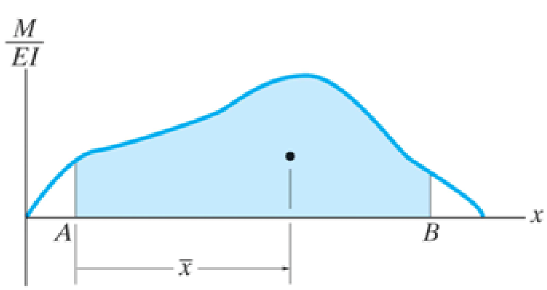




Notes:

- If the moment is positive $(+)$, the tangent at A is above the extended tangent
- If the moment is negative $(-)$, the tangent at A is below the extended tangent
- Note $t_{A/B} \neq t_{B/A}$





It is important to note that both $t_{A/B}$ and $\theta_{B/A}$ do not directly give slope and deflection (recall they are measured with respect to tangents)

- One must use both $\theta_{B/A}$ and $t_{A/B}$ along with knowledge of the shape of the elastic curve to be able to find the slope and deflection we need.

Usually it is helpful to draw the tangents at support locations since these will usually have zero deflection or slope …

pin or roller support $\Delta = 0$

Δ = deflection
θ = rotation

fixed support $\Delta = 0$ and $\theta = 0$

Procedure for analysis

1. Determine the reactions
2. Draw the moment diagram
    - Draw the M/EI diagram
3. Draw an exaggerated elastic curve of the beam
4. Apply theorem 1 to determine the angle between any two tangents
5. Apply theorem 2 to determine tangential deviations
    - Most times theorem 2 **won't** give you the desired deflection directly

Useful unit conversions:

- 1 ft = 12 in
- 1 $ft^2 = 12^2\ in^2$
- 1 $ft^3 = 12^3\ in^3$
- 1 $ft^4 = 12^4\ in^4$

- 1 m = 1 x 10^3 mm
- 1 m^2 = 1 x $10^6\ mm^2$
- 1 m^3 = 1 x $10^9\ mm^3$
- 1 m^4 = 1 x $10^{12}\ mm^3$
- 1 GPa = 1 x $10^6\ kN/m^2$

3. Conjugate beam method

This method was derived in the late 1800s by H. Muller-Breslau.

- It is a graphical technique for finding slope and deflection.
- It is somewhat simpler to use than the moment-area method since it only relies on a proper knowledge of statics.

The method is based on the relationships for loading, shear and moment that we saw previously.

The following relationships relate internal shear and moment to the applied load

$$\frac{dV}{dx} = -w(x) \qquad \frac{dM}{dx} = V(x) \rightarrow \frac{d^2M}{dx^2} = -w(x)$$

$$V = -\int w(x)dx \qquad M = -\int\int w(x)dx\,dx$$

The following relationships relate slope and deflection to the internal moment:

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad y = \int\int\left(\frac{M}{EI}\right)dx\,dx$$

Notice anything similar?

The relationship for **SLOPE** compares to the relationship for **SHEAR**

The relationship for **DEFLECTION** compares to the relationship for **MOMENT**

1)	**Load**	*is to*	**Shear**	*as*	**Curvature**	*is to*	**Slope**
	$w(x)$	$\rightarrow$	$V(x) = -\int w(x)dx$	:	$\frac{M}{EI}(x)$	$\rightarrow$	$\theta(x) = \int \frac{M}{EI}(x)dx$
2)	**Load**	*is to*	**Moment**	*as*	**Curvature**	*is to*	**Deflection**
	$w(x)$	$\rightarrow$	$M(x) = -\iint w(x)dx$	:	$\frac{M}{EI}(x)$	$\rightarrow$	$v(x) = \iint \frac{M}{EI}(x)dx$

The method we will see next takes advantage of these similarities....

FIRST: We draw a "conjugate beam"

- we choose appropriate supports for the conjugate beam to ensure:
    - .... the **shear** **in the <u>conjugate beam</u>** corresponds to the **slope** **in the <u>real beam</u>**
    - .... the **moment** **in the <u>conjugate beam</u>** corresponds to the **displacement** **in the <u>real beam</u>**

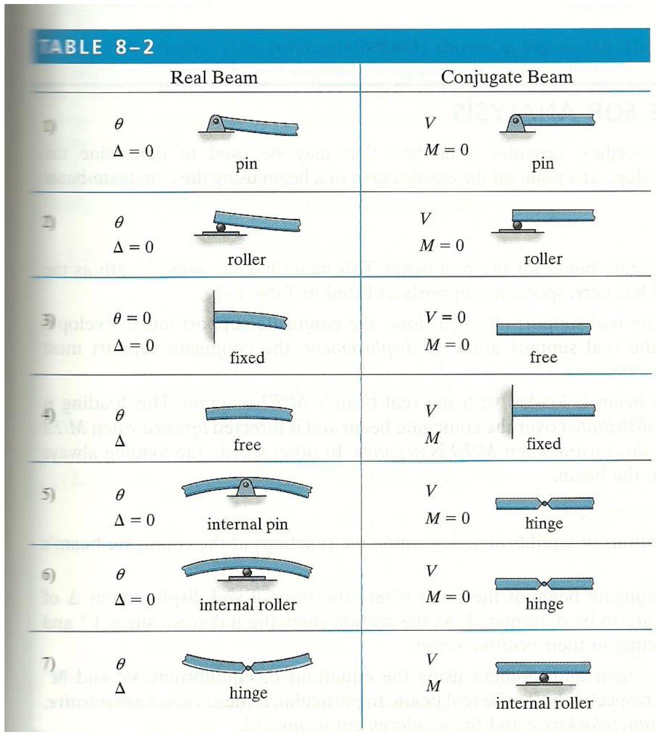
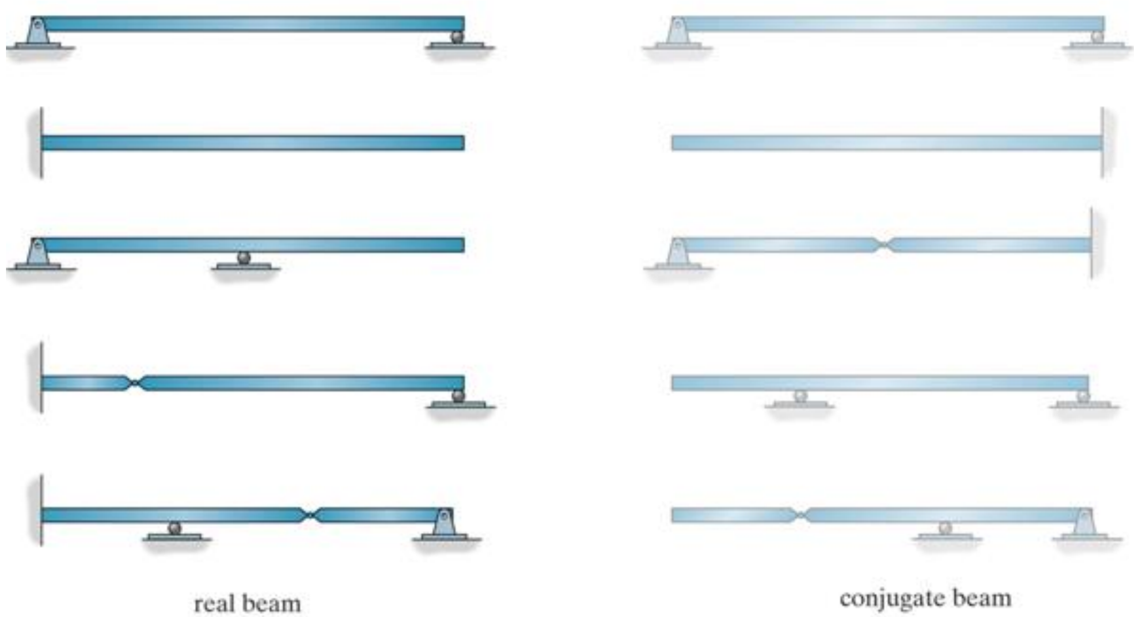


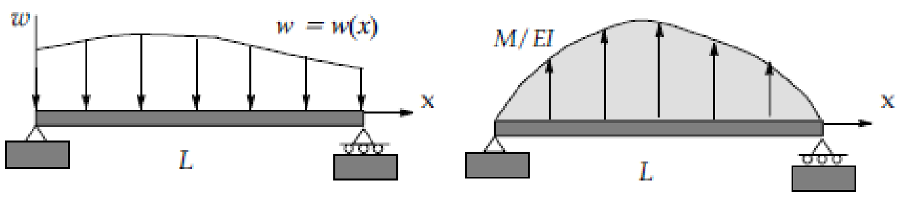
TABLE 8–2

	Real Beam		Conjugate Beam	
1)	θ, $\Delta = 0$	pin	V, $M = 0$	pin
2)	θ, $\Delta = 0$	roller	V, $M = 0$	roller
3)	$\theta = 0$, $\Delta = 0$	fixed	$V = 0$, $M = 0$	free
4)	θ, Δ	free	V, M	fixed
5)	θ, $\Delta = 0$	internal pin	V, $M = 0$	hinge
6)	θ, $\Delta = 0$	internal roller	V, $M = 0$	hinge
7)	θ, Δ	hinge	V, M	internal roller

Examples - real beam vs. conjugate beam



NEXT: The conjugate beam is loaded with the M/EI diagram (simulates "w(x)")



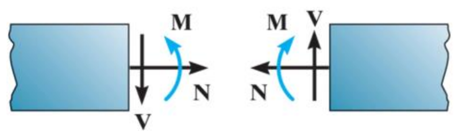
NEXT: We apply the two conjugate beam theorems:

(Theorem1) The slope at a point in the real beam is equal to the shear at the corresponding point in the conjugate beam

(Theorem2) The deflection at a point in the real beam is equal to the moment at the corresponding point in the conjugate beam

Procedure for analysis

1. Determine the reactions
2. Draw the moment diagram
   - Draw the M/EI diagram
3. Draw the conjugate beam with appropriate supports
4. Load the conjugate beam with a loading corresponding to the M/EI diagram of the real beam (as distributed load)
   - **hint**: just draw loading to be always acting **away** from the beam !
5. Analyze conjugate beam: determine reactions at supports of CJ-beam
6. **section** the CJ-beam at the point where **slope** and **displacement** of the real beam are to be determined
   - at the section show V' and M' acting in positive sense



   - Apply theorem 1 to determine shear ..... V' = ϴ
     - If V' = positive, ϴ is **counterclockwise**
   - Apply theorem 2 to determine moment ...... M' = Δ
     - If M' = positive, Δ is **upwards**