Lecture 4.1: Virtual Work - Trusses

Lecture outline:

1. Energy methods for computing deflections
2. Principal of virtual work
3. Virtual work: trusses
4. Procedure

1. Energy methods for computing deflections:

Energy methods are based on the principal of conservation of energy:

$$U_e = U_i$$

***work** done by all external forces acting on a structure,* U_e

= *internal work (strain energy stored in the structure)* , U_i

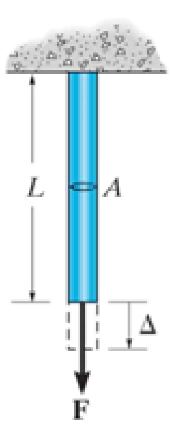
External work due to a Force (U_e)

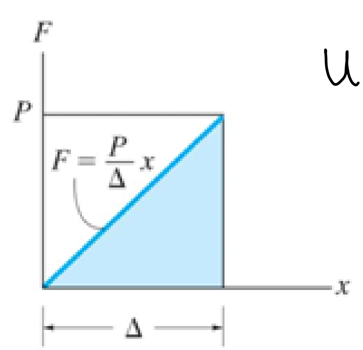
When a force, *F*, displaces an amount *dx* in the same direction as the force the total work done is:

$$dU_e = Fdx \rightarrow U_e = \int_0^x Fdx$$

Work = Area under the force-displacement diagram

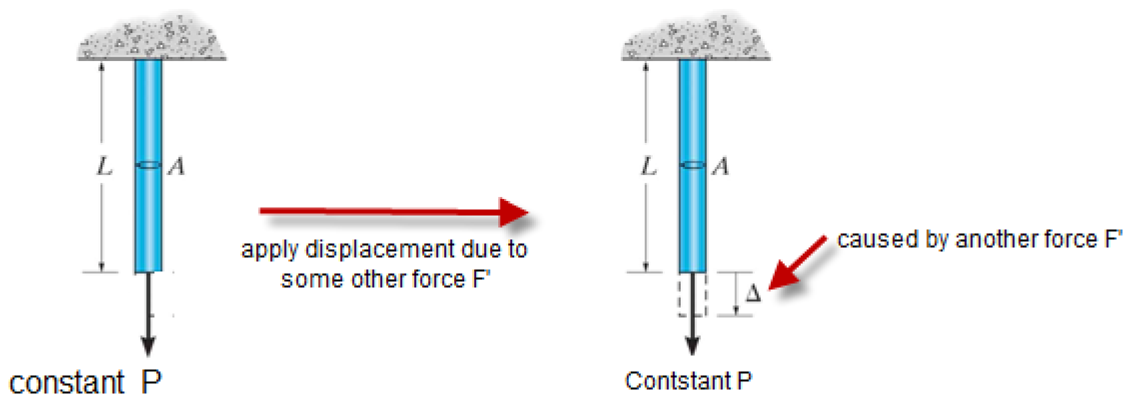
If the force is gradually applied to a linearly elastic structure we may express the variation of force with displacement as





$U_e = P\frac{\Delta}{2}$

Another case that we are interested in is the work done by a constant force while its point of application undergoes a displacement caused by some other force independent of P:



Internal work (U_i)

Strain energy due to an axial force: When an axial force, N, is applied to a linear elastic bar, the external work done by the bar will be converted into strain energy.

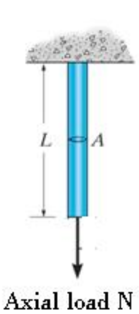
The **stress** in the bar is given by: $\sigma = \dfrac{N}{A}$ Where A is the cross-sectional area of the bar.

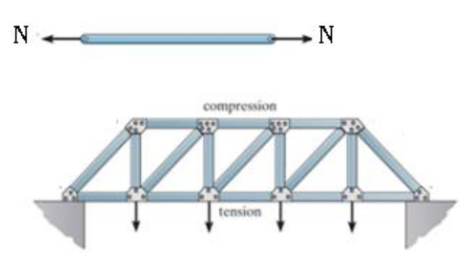
The **final strain** of the bar is : $\varepsilon = \dfrac{\delta}{L}$ Where ε is the strain and δ is the final displacement.

If the bar is linear elastic then Hooke's law will be valid: $\sigma = E\varepsilon$

Solving for the internal displacement: $\delta = \dfrac{NL}{AE}$

Substituting into the equation for work: $U_i = \dfrac{1}{2} N\delta = \dfrac{1}{2} N \dfrac{NL}{AE}$





2. Principal of virtual work

The principle of virtual work is based on the conservation of energy and was developed by John Bernoulli in 1717 (also called **unit load method**)

It is a general procedure for determining the displacement and slope at any point in a structure (trusses, beams and frames).

If we apply a series of external forces P to a structure:

- it will cause a set of internal forces, N, through the structure.

Due to the loading:

- EXTERNAL displacements Δ will occur due to EXTERNAL loads **P**
- INTERNAL displacements δ will occur due to INTERNAL loads **N**

$$\sum(P*\Delta)=\sum(N*\delta)$$

$$\text{Work of External loads} = \text{Work of Internal loads}$$

As luck would have it:

- point where we typically want to find Δ ≠ location of external loads P (usually)

In order to calculate displacement anywhere in a structure we can use the concept of a "**virtual**" load. "**Virtual**" means that the load is imaginary and does not exist as part of the real loading system.

Place **P'** in the direction of the unknown displacement

P' causes a set of internal virtual forces → n

Next the structure is subjected to the real loading (P):

- causes real displacement Δ
- causes real internal forces N → real internal displacements δ
- The internal virtual loads n move through the real internal displacements δ

The external work must be equal to the internal work so we can write:

virtual loadings

$$P' \cdot \Delta = \sum n \cdot \delta$$

real displacements

By choosing **P'** =1 we end up with:

$$1 \cdot \Delta = \sum n \cdot \delta$$

3. Virtual work method: Trusses

We can apply the above equation for solving the displacement in a truss.

If the applied load causes linear elastic deformations, the **internal** displacements → $\delta = \frac{NL}{AE}$

Therefore the virtual force equation $1 \cdot \Delta = \sum n \cdot \delta$ becomes:

$$\boxed{1 \cdot \Delta = \sum n \cdot \frac{NL}{AE}}$$

- 1 = external **virtual** unit load
- n = internal force in truss member caused by the external **virtual** force

- Δ = external displacement due to **real** loads
- N = internal force in truss member caused by the external **real** loads

- L , A , E = length, cross-section area & modulus of elasticity of a member

Temperature effects:

In some cases trusses members may change their length due to temperature:

- α is the coefficient of thermal expansion for a member\
- ΔT is the change in temperature
- $\Delta L = \alpha \Delta T L$ is the resulting change in member length

To determine the displacement at a truss joint due to this temperature change we use:

$$1 \cdot \Delta = \sum n \cdot \alpha \Delta T L$$

- if a member undergoes an **increase** in temperature, ΔT will be **positive (+)**
- if a member undergoes an **decrease** in temperature, ΔT will be **negative (-)**

Fabrication errors:

In some cases trusses members may be made longer or shorter either due to fabrication errors:

- ΔL is the resulting change in member length

If a truss member is shorter or longer than expected the resulting displacement:

$$1 \cdot \Delta = \sum n \cdot \Delta L$$

- if a fabrication error **increases** the length of a member: ΔL will be **positive (+)**
- if a fabrication error **decreases** the length of a member: ΔL will be **negative (-)**

In a problem with all these effects → $1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$

4. Procedure

Procedure - VW for trusses:

1. Ensure that truss is determinate and stable:
   - Recall if : $b + r = 2j$ → truss is determinate

2. Structure subjected to real forces:
   - Use method of sections/joints to determine internal forces due to **real** loads:
     - get "N" values

3. Structure subjected to virtual force:
   - Remove real loads
   - Place a unit load (= 1 kN) on the truss at the joint where the displacement is to be determined
   - Load should be in same direction as the displacement
   - Use method of sections/joints to determine internal forces due to virtual loads
     - get "n" values
     - assume that T = positive, C = negative

4. Apply virtual-work equation:

$$1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$$

- if a member undergoes an **increase** in temperature, ΔT will be **positive (+)**
- if a fabrication error **increases** the length of a member: ΔL will be **positive (+)**

General notes:

- Be careful to retain the algebraic sign for each of "n'" and "N"
  - If right hand-side is positive (+) : displacement is in same direction as unit load
  - If left hand-side is negative (-) : displacement is in opposite direction as unit load
- Be careful to be consistent with units
  - GPa = 1×10^6 kN/m^2
  - mm^2 = 1×10^{-6} m^2

Lecture 4.2: Virtual work - Beams

Lecture outline:

1. Principal of virtual-work for beams/frames
2. Procedure
   - Option 1: Direct Integration
   - Option 2A: Evaluating $\int_b^a mMdx$ using graphical method
   - Option 2B: Evaluating $\int_b^a mMdx$ using Mohr's integration tables
3. Integration by parts
4. Other effects: axial, shear, torsion and temperature effects

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1. Principal of virtual work: beams/frames

The principal of virtual work we discussed for trusses can also be used to find the deflection of beams. Deflection in a beam is primarily due to strain caused by bending.

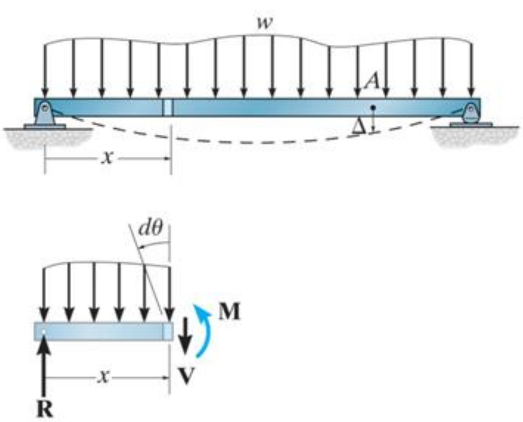
Real loads → Real internal moment, M → real displacement Δ.

Real deflection is at "a" = Δ + Real internal rotation is $d\theta$.

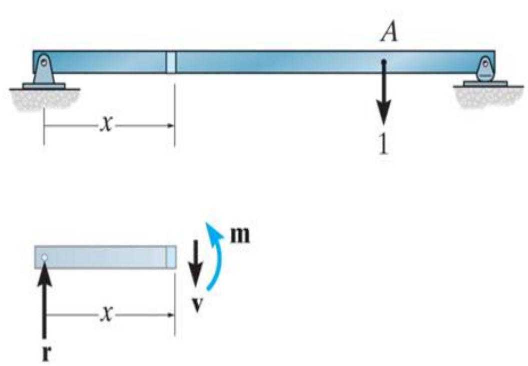
Next, apply a virtual load (**P'** =1) @ point where Δ is required→ internal virtual moment, m

External work = $1 * \Delta$ Internal virtual work = $m * d\theta$

$$1 \cdot \Delta = \sum m \cdot d\theta$$



real loads



virtual loads

Deflection/Displacement:

Recalling that the moment is related to the slope by $d\theta = \frac{M}{EI}dx$,

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

- 1 = external **virtual** unit load applied at point "a"
- m = internal moment caused by the external **virtual** force

- Δ_a = deflection at point "a' due to **real** loads
- M = internal moment caused by the external **real** loads

- E, I = Modulus of elasticity of & moment of inertia of the beam cross-section

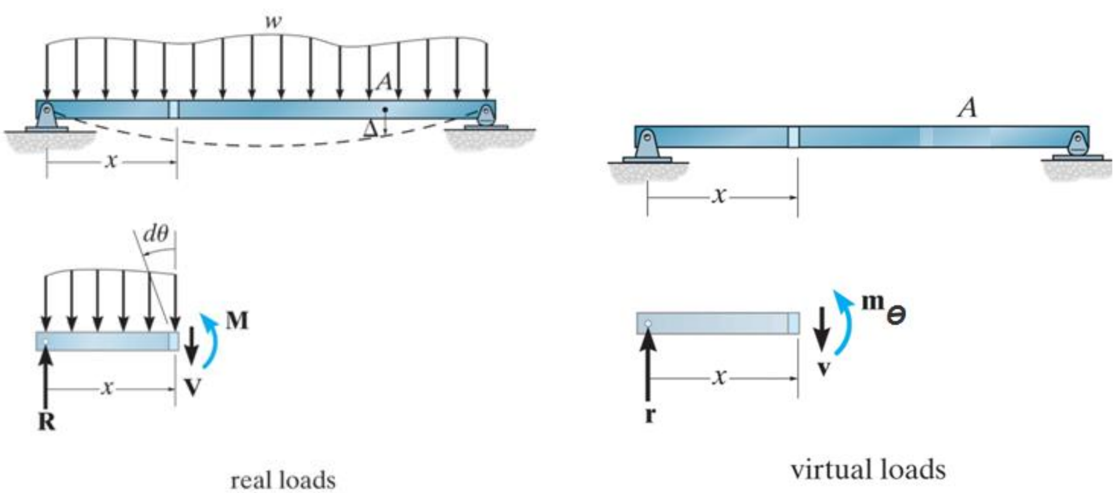
Slope:

In a similar manner, the **slope** at a point may be obtained using:.

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

- 1 = external **virtual** unit couple applied at point "a"
- θ_a = slope at point "a' due to **real** loads
- m_θ = internal moment caused by the external **virtual** couple
- M = internal moment caused by the external **real** loads
- E, I = Modulus of elasticity of & moment of inertia of the beam cross-section

Notice that the expression is the same except that we now apply a *virtual* **couple** at the point in question where the slope is desired.



2. Procedure for analysis

Procedure:

1. Ensure that beam/frame is determinate and stable:
   - Recall if : $3m + r = 3j + e_c$ → beam/frame is determinate
2. Structure subjected to real forces:
   - With the **real** loads on the structure, use statics to determine internal moments *M*.
     - It may easier to treat different external loads separately and then add their effects using the principle of superposition
3. Structure subjected to virtual force:
   - Remove real loads
   - If **displacement** is desired:
     - Place a unit load (= 1 kN) at the point where the desired displacement is to be determined
   - If **slope** is desired:
     - Place a unit couple (= 1 kN m) at the point where the desired slope is to be determined
   - Calculate the internal moments m or m_θ due to the virtual load alone.
4. Apply virtual-work equation:

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx \quad \text{or} \quad 1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

   - Be careful to retain the algebraic sign for each of "m'" and "M"
   - Be careful to be consistent with units
     - $mm^2 = 1 \times 10^{-6}\ m^2$
     - $GPa = 1 \times 10^{6}\ kN/m^2$
     - $mm^4 = 1 \times 10^{-12}\ m^4$
   - If the sum of the integrals is:
     - positive (+) : displacement/slope is in same direction as unit load/couple
     - negative (-) : displacement is in opposite direction as unit load/couple

Note: There are three options t**o evaluate** $\int_b^a mM dx$

- **Option 1 : use Direct Integration ... i.e. actually solving the integrals mathematically**
- **Option 2A: Use Graphical method** **... see p.4-5**
- **Option 2B: Use Mohr's Integration tables** **... see p.6-7**

Option 2A: using **Graphical method to evaluate** $\int_b^a mM dx$

Rather than computing the integral using integration one can use a graphical method which is somewhat simpler and less time consuming.

Procedure:

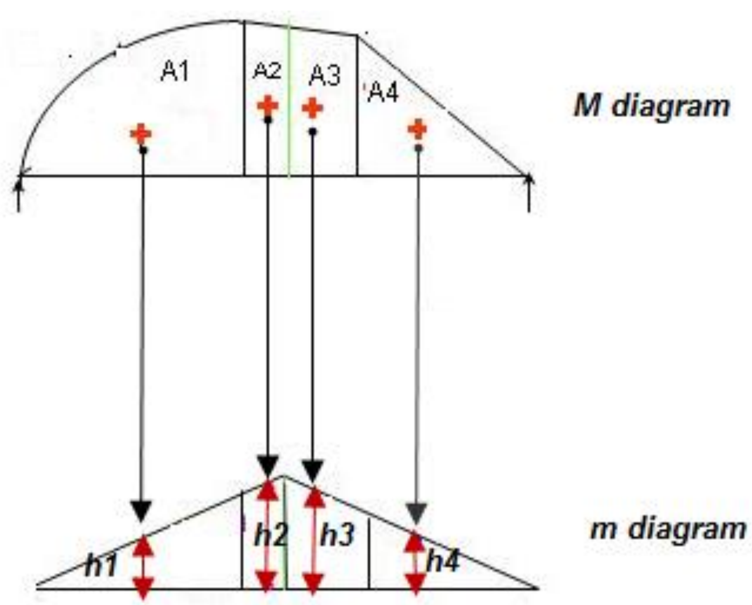
1) Draw the ***M*-diagram** for applied loading
    a. if there is multiple loads, draw the M diagram by "parts"

2) Draw the ***m*-diagram** due to the unit virtual load/couple

3) Divide the ***M*-diagram**, into segments[1] for which you can easily find the area (e.g. triangle, rectangle, parabola …)

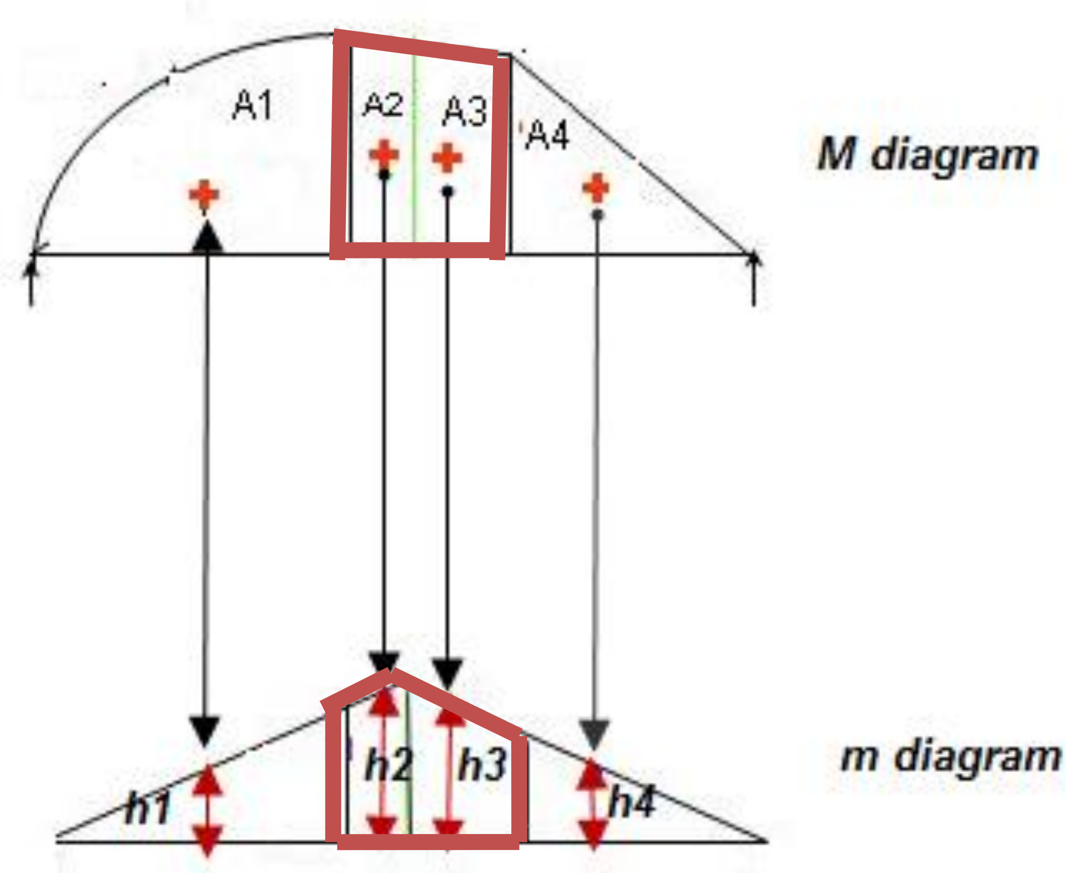
4) For each segment:
    a. Calculate the area A_i on the ***M*-diagram**
    b. Locate the center of gravity CG_i on the ***M*-diagram**
    c. Project the location of CG_i onto the ***m*-diagram**
    d. Pick up the height, h_i, on the ***m*-diagram**
    e. Compute $A_i * h_i$ for each segment

5) To get $\int_0^L \frac{mM}{EI} dx$, sum up the results for all the segments: $\frac{1}{EI} \sum_i A_i * h_i$



Note

- Both moment diagrams must be continuous over each segment length.
- If ***M*** or ***m*** diagram is not continuous, one MUST divide into more segments, each of which is continuous over the integration length



Option 2B: using Mohr's **integration tables to evaluate** $\int_b^a mM dx$

As an alternative we can use "Mohr" integration tables to evaluate the integral $\int_0^L mM dx$,

recognizing that it is of the format $\int_0^L f_1(x) * f_2(x) dx$.

Procedure:

1) Complete steps 1-3 as before

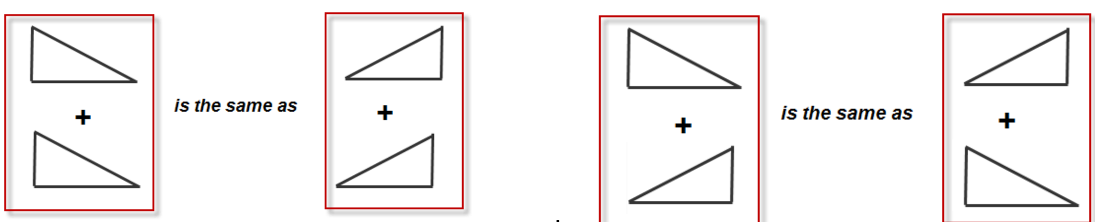
2) Take the row value $f_1(x)$ that corresponds to either the "M" or "m" segment

3) Take the column value $f_2(x)$ that corresponds to either the "m" or "M" segment

4) Compute the area in the cell corresponding to the $f_1(x)$ & $f_2(x)$ combination

5) To get $\int_0^L \frac{mM}{EI} dx$, sum up the results for all the segments !

Note #1:



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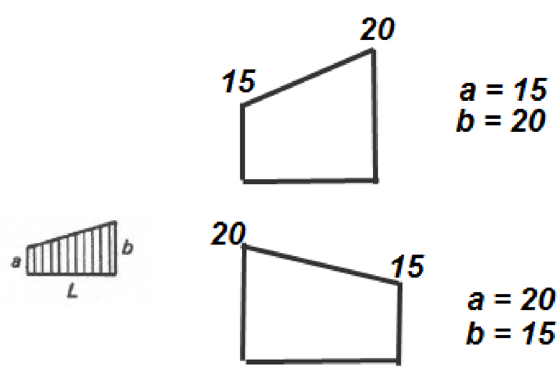
Note #2:



MOHR'S INTEGRATION TABLES

Note: Select a **f1(x)** and the matching **f2(x)** ... order doesn't matter ... i.e. f1(x) can be the row or column

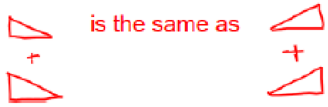
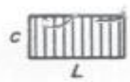
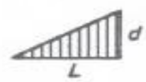
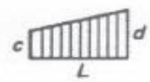
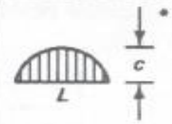
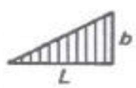
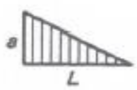
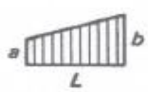
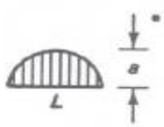
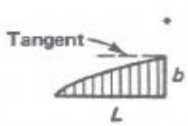
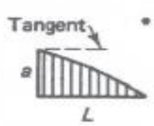
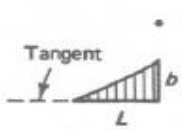
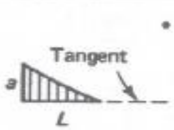
Note: 

TABLE 7.1 VALUES OF $\int_0^L f_1(x) f_2(x)\,dx$

$f_1(x)$ \ $f_2(x)$				
	$\frac{1}{2} Lbc$	$\frac{1}{3} Lbd$	$\frac{Lb}{6}(c + 2d)$	$\frac{1}{3} Lbc$
	$\frac{1}{2} Lac$	$\frac{1}{6} Lad$	$\frac{La}{6}(2c + d)$	$\frac{1}{3} Lac$
	$\frac{L}{2}(a + b)c$	$\frac{Ld}{6}(a + 2b)$	$\frac{L}{6}(2ac + ad + 2bd + bc)$	$\frac{Lc}{3}(a + b)$
	$\frac{2}{3} Lac$	$\frac{1}{3} Lad$	$\frac{La}{3}(c + d)$	$\frac{8}{15} Lac$
	$\frac{2}{3} Lbc$	$\frac{5}{12} Lbd$	$\frac{Lb}{12}(3c + 5d)$	$\frac{7}{15} Lbc$
	$\frac{2}{3} Lac$	$\frac{1}{4} Lad$	$\frac{La}{12}(5c + 3d)$	$\frac{7}{15} Lac$
	$\frac{1}{3} Lbc$	$\frac{1}{4} Lbd$	$\frac{Lb}{12}(c + 3d)$	$\frac{1}{5} Lbc$
	$\frac{1}{3} Lac$	$\frac{1}{12} Lad$	$\frac{La}{12}(3c + d)$	$\frac{1}{5} Lac$
$\int f_2^2(x)dx$	Lc^2	$\frac{1}{3} Ld^2$	$\frac{L}{3}(c^2 + cd + d^2)$	$\frac{8}{15} Lc^2$

* Second-degree parabola.

3. Drawing M diagrams by parts:

Using superposition:

Superposition can be helpful if we plan to use Mohr's tables since if we have different loadings it can lead to complicated moment diagrams , and if the diagram is complicated we will not be able to find the combination of M *m on the Mohr's table sheet.